Spanning Tree Transformation of Connected Graphs into Single-Row Networks

S.L. Loh, S. Salleh, and N.H. Sarmin

Abstract—A spanning tree of a connected graph is a tree which consists of the set of vertices and some or perhaps all of the edges from the connected graph. In this paper, a model for spanning tree transformation of connected graphs into single-row networks, namely Spanning Tree of Connected Graph Modeling (STCGM) will be introduced. Path-Growing Tree-Forming algorithm applied with Vertex-Prioritized is contained in the model to produce the spanning tree from the connected graph. Paths are produced by Path-Growing and they are combined into a spanning tree by Tree-Forming. The spanning tree that is produced from the connected graph is then transformed into single-row network using Tree Sequence Modeling (TSM). Finally, the single-row routing problem is solved using a method called Enhanced Simulated Annealing for Single-Row Routing (ESSR).

Keywords—Graph theory, simulated annealing, single-row routing and spanning tree.

I. INTRODUCTION

Any engineering and science problems can be represented as a problem in graph theory. The scenario of the real-life applications is represented by the graph where the nodes in the graph can be treated as nodes in a network, and the communication links between the nodes are represented by the edges. To solve such a problem, the graph which represents the problem is transformed into single-row network where it is somehow the complexity of solution computations is reduced and solved as a single-row routing problem.

Single-row routing problem is an NP-complete problem. The necessary and sufficient condition for optimum single-row routing had been developed by Ting et al. [1] and Kuh et al. [2]. Many researches had been done to solve single-row routing problem such as the use of some heuristic algorithms [3] and applied graph theoretical representation [4]. The main objective for all these researches is to minimize the congestion. A model which minimizes both the congestion and the number of doglegs was developed, namely Enhanced Simulated Annealing for Single-Row Routing (ESSR) [5]. Congestion and doglegs in the single-row network are represented by the total energy, or $E$, which is the fitness value of the model.

The researches on transformation from connected graphs into single-row networks had been done for the first time in [6] and [7]. The optimality in nodes labeling was tackled in [8] based on simulated annealing [9]. The effort was extended from [8] to the single-row transformation of complete binary tree and the tree [10]. After the single-row transformation, the single-row routing problem is solved using ESSR.

In this work, the main concern is the spanning tree transformation of connected graph which can lead to the optimization of the single-row transformation. The connected graph is assumed to have one unit of weight for each edge. The relationship between connected graph with its spanning tree and the spanning tree with its single-row representation were studied in order to develop an efficient modeling which can optimize the spanning tree transformation of connected graph into single-row network.

The spanning tree of a connected graph is a tree that consists of all vertices and some (or perhaps all) of the edges from the connected graph. It can also be defined as the minimal set of edges that connects all the vertices. We found that the spanning tree transformation of a connected graph into the single-row network is optimal when that spanning tree has a set of nodes with minimum highest degree. In other words, the spanning tree has the fewest leaves. A model is developed based on this finding, namely Spanning Tree of Connected Graph Modeling (STCGM). The spanning tree which produced from the connected graph will be transformed into single-row network by Tree Sequence Modeling (TSM) [11] before ESSR is applied to solve the single-row routing problem.

This paper is organized into seven sections. Introduction is presented in Section 1 while Problem background is stated in Section 2. The relationship between connected graph, spanning tree and its single-row network is shown in Section 3. The fundamental concepts and findings from the study on relationship between a spanning tree from a connected graph and its single-row representation are included in Section 4. The developed model for spanning tree transformation of connected graphs into single-row networks, namely Spanning

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Tree of Connected Graph Modeling (STCGM) is shown in Section 5. The experimental results are presented in Section 6. Summary and conclusion are consisted in Section 7 while Section 8 is the acknowledgements.

II. PROBLEM BACKGROUND

The problem can be stated as follows: Given a connected graph, a spanning tree is produced from the connected graph and it needs to be transformed into single-row representation.

The problem is illustrated in Figure 1. A connected graph \( G \), the spanning tree \( SP \) from the \( G \) and the single-row representation \( S \) of the spanning tree are shown in Figure 1(a), (b) and (c) respectively.

![Figure 1](image)

Fig. 1 (a) A connected graph \( G \), (b) spanning tree from the \( G \) and (c) its single-row representation \( S \).

III. RELATIONSHIP BETWEEN CONNECTED GRAPH, SPANNING TREE AND ITS SINGLE-ROW NETWORK

In the spanning tree transformation of connected graphs into single-row networks, a spanning tree \( SP \) is produced from the given connected graph \( G \) before it is transformed into single-row network. The spanning tree is a tree which composed all of the vertices and some or maybe all of the edges from the connected graph. A spanning tree \( SP \) contains \((n-1)\) edges. There are many forms of spanning trees that can be produced from the connected graph by selecting different sets of edges from \( G \). In order to optimize the transformation from (a) to (c) as shown in Figure 1, the necessary criterion in producing the optimal spanning tree is the course of this research.

In a connected graph, \( G \), there are \( n \) nodes where each node \( v_i \) has \( d_i \) degree. After the spanning tree is produced from the connected graph, it will be transformed into single-row representation as shown in Figure 1(c). Every node \( v_i \) in \( SP \) is transformed into a zone \( z_i \) for \( i = 1, 2, ..., n \), in the single-row network \( S \). Each \( v_i \) has \( D_i \) degree in \( SP \) and thus each \( z_i \) has \( D_i \) number of terminals which evenly spaced aligned on node axis in \( S \). The communication links in \( SP \) are preserved where the edges are transformed into nets in \( S \). The nets are made up of horizontal and vertical line segments which are drawn from left to right without crossing or backward movement. Congestion, or \( Q \), is the maximum total number of horizontal tracks between upper and lower street congestion. A vertical crossing in node axis is called a dogleg, or \( D \). The sequence of zones and the nets are arranged in such a way the congestion and dogleg are being minimized.

IV. FUNDAMENTAL CONCEPTS

A. G-SP

In this research, the connected graph \( G \) is assumed to have weight of one unit for each of the edges. Hence, there are various forms of spanning tree \( SP \) that can be formed from the connected graph. Now the question is: how is the form of \( SP \) that can optimize the single-row transformation? To obtain the answer, backward searching is needed where it starts with the study of relationship between \( SP \) and \( S \) before \( G \) and \( SP \). In short, the form of \( SP \) which can lead to the optimal single-row transformation from \( SP \) to \( S \) is needed to be determined before the \( G \) is transformed into \( SP \).

B. G-SP

A spanning tree \( SP \) can be treated as a tree. A research had been carried out in studying the relationship between the tree and its single-row representation. A tree with \( n \) vertices gives the maximum value of \( E \), \( Q \) and \( D \) results in single-row network when the tree is in the form of 2-Level \( T \). The optimum result for 2-Level \( T \) is \( E = Q(n-1-2Q) \) where \( Q = \left[ \frac{(n-3)}{4} \right] \) and \( D = 0 \). Proposition 1 is developed from 2-Level \( T \)'s results to find out the necessary criteria of spanning tree which optimizes the spanning tree transformation of connected graph into single-row network.

C. Proposition

By having the same total number of nodes, a spanning tree with smaller value of \( D_{max} = max\{D_i\} \) gives the lower congestion for single-row transformation of spanning tree.

**Proof:**

A spanning tree \( SP \) has a total degree \( D \) where \( D = \sum_{i=1}^{n} D_i = 2(n-1) \). The spanning tree \( SP \) can be treated
as a tree $T_n$ and the highest degree among the nodes in the $SP_n$ denoted by $D_{max} = \max_{i \leq n} \{ D_i \}$. Different sets of $D_i$ for $i = 1, 2, ..., n$, present the different structures of $SP_n$. For every $D_i$, the corresponding $z_i$ has $D_i$ number of nets to be formed. As a minimum, the $z_i$ will contribute to the total congestion \( Q = \left[ (\frac{D_i - 1}{2})^2 + 1 \right] \) in $S_n$ by replacing $D_i = n - 1$ in the optimum result from 2-Level $T_n$. Obviously, the larger value of $D_i$ will lead to the larger value of total congestion.

Thus, $D_{max}$ among the $D_i$ affects the result significantly where the set of $D_i$ which gives the smallest value of $D_{max}$ will contribute to the lower congestion in single-row transformation. The optimum result for the optimum set of $D_i$ is $E = Q = D = 0$ given by $D_{max} = 2$ where it is exactly a Hamiltonian path.

D. G-S

In the spanning tree transformation of $G_n$ into single-row network, the result is optimum ($E = Q = D = 0$) when the spanning tree $SP_n$ produced from $G_n$ is a Hamiltonian path. A graph that contains a Hamiltonian path is called the traceable graph. For the connected graph $G_n$ other than the traceable graph, the $SP_n$ that produced from $G_n$ need to be the $SP_n$ with the minimum value of $D_{max}$. By having the same total number of nodes, the connected graph with larger number of edges tends to give a Hamiltonian path or a spanning tree with fewer leaves compared with the connected graph with lower number of edges. Spanning tree $SP_n$ with minimum value of $D_{max}$ minimizes the total energy value and congestion as well as doglegs in single-row network.

V. OUR STCGM MODEL

From a connected graph, a spanning tree is produced and being transformed into single-row representation. For optimal spanning tree transformation of connected graph into single-row network, the necessary requirement is the spanning tree produced from the connected graph has the minimum value of $D_{max}$. To obtain the $SP_n$ with minimum value of $D_{max}$, the Path-Growing Tree-Forming algorithm is developed to produce the $SP_n$ from the $G_n$.

A. Path-Growing Tree-Forming Algorithm

Obviously, Path-Growing Tree-Forming algorithm is made up of two algorithms. Path-Growing algorithm produces the paths while Tree-Forming algorithm connects all the paths that are formed by Path-Growing algorithm into a spanning tree with the minimum value of $D_{max}$. Vertex-Prioritized is applied in Path-Growing Tree-Forming algorithm to minimize the $D_{max}$ in the $SP_n$.

Initially, notation $C_{ik}$ is defined as the set of nodes of the $SP_n$ and notation $C_{2k}$ is defined as the set of nodes of the path while notation $\bar{C}_k$ denotes the set of unconnected nodes at $k^{th}$ iteration. In the Path-Growing Tree-Forming algorithm, Vertex-Prioritized is applied where the node with the least degree is given the highest priority to be chosen. Notation $c_{-d}$ represents the degree of nodes in set $C_{ik}$ for $SP_n$ while $\bar{c}_{-d}$ represents the degree of nodes in set $\bar{C}_k$ where it is involving only the edges connected by the nodes in the set itself.

First of all, a path is formed by Path-Growing algorithm. The node with the least degree in set $\bar{C}_k$ is chosen to be the initial node $i^*$ for the path to be formed. This step enables the possible of obtaining the path with maximum length. Once the node is included in the path construction, it will be moved from set $\bar{C}_k$ to set $C_k$. In Path-Growing, when the path is impossible to be connected to the tree by any edge in length of one unit, Backward Tracing will be applied in Path-Growing and a walk will be produced instead of a path.

The path is expanded by choosing node $j^*$ from $\bar{C}_k$ under Vertex-Prioritized in which it is connected with the node $i^*$. The node $i^*$ is updated as node $j^*$ whenever a node $j^*$ is found. The first formed path acts as the spanning tree. When there is no such a node $j^*$ is found in $\bar{C}_k$, the next path need to be produced from set $\bar{C}_k$ if set $\bar{C}_k$ is not yet a null set. The process is repeated for Path-Growing algorithm applied with Vertex-Prioritized to form another path. Tree-Forming algorithm is developed to join the new formed path to the spanning tree by joining the new formed path to the node with the least $c_{-d}$ from the spanning tree.

Here is the algorithm for Path-Growing Tree-Forming.

\[ C_{ik} = \text{set of nodes of spanning tree at iteration } k. \]
\[ C_{2k} = \text{set of nodes of path at iteration } k. \]
\[ \bar{C}_k = \text{set of unconnected nodes.} \]
\[ c_{-d} = \text{current degree of every node in } C_k. \]
\[ \bar{c}_{-d} = \text{current degree of every node in } \bar{C}_k. \]
\[ E_k = \text{set of edges of the spanning tree.} \]

Set $C_{00} = \emptyset$, $C_{20} = \emptyset$, $\bar{C}_0 = N$ and $a = 0$;

\[ \text{for } i = 1 \text{ to } n \]
\[ c_{-d} = 0; \]
\[ \bar{c}_{-d} = d_i; \]

Select the node with least $\bar{c}_{-d}$, set $C_{2i} = \{ i \}$, set $i^* = i$ and $\bar{C}_i = N - \{ i \}$;

\[ \text{for } k = 1 \text{ to } n - 1 \]
Identify the unconnected node \( j^* \) in the \( \overline{C}_k \)
that has the least \( \overline{c}_{-d} \) which connected with
node \( i^* \) in \( C_{2k} \); if such \( j^* \) is found
Connect these two nodes:
\[
E[i^*][j^*] = 1;
\]
Set \( i^* = j^* \);
/*Update degree of nodes in \( C_k \) and \( \overline{C}_k \)*/
\[
c_{-d}_{i^*}++; \quad \overline{c}_{-d}--; \quad \text{for nodes in } \overline{C}_k \text{ which}
\quad \text{connected (in } \mathcal{G}_n \text{) with node } i^* \text{ in }
\quad C_{2k} ;
\]
//Check connection of path to tree
Identify the connected node \( j'' \) in
the \( kC_1 \) that has the least \( c_{-d} \) that
connected with node \( i^* \) in \( C_{2k} \);
Set \( i'' = i^* \);
else
\[
a++; \quad \text{if } a = 1 \text{ or } (i'' \text{ and } j'' \text{ are found})
\]
Set \( C_{ik} = C_{2k} \);
if \( a > 1 \) //Connect the path formed
with the spanning tree
\[
\text{if } i'' \text{ and } j'' \text{ are found}
\]
Connect these two
nodes:
\[
E[i''][j''] = 1;
\]
Update degree of nodes
in \( C_k \) and \( \overline{C}_k \)
\[
k++; \quad \text{else}
\]
Backward Tracing to get the
\[
i^* ;
\]
\[
k--; \quad \text{if } i'' \text{ and } j'' \text{ are found}
\]
//Select a node to form a path
Select the node with least
\[
\overline{c}_{-d}, \quad \text{set } i^* = i ; \quad \text{Check connection of path to}
\quad \text{tree}; \quad \text{if } \overline{C} \neq 0
\]
\[
k--; \quad \text{if } \overline{C} \neq 0
\]

B. Tree Alteration for Single-Row Transformation

For illustration, after the spanning tree \( SP_9 \) is produced from
connected graph \( G_n \) which is shown in Figure 2, the \( SP_n \) will
be transformed into a rooted tree as shown in Figure 3(b) and
the rooted tree is solved using Tree Sequence Modeling (TSM).

Fig. 2 A connected graph, \( G_9 \)

Fig. 3 The two depictions of the output ( \( SP_9 \) ) from Path-Growing Tree-Forming

From Figure 3(b), node \( v_2 \) is chosen to be the root node as
it has the highest degree \( D_{\max} \) among the nodes in the \( SP_9 \).
Actually, the results are slightly different based on the chosen
node for the root node. However, this problem can be
overcome by TSM where it is a specifically developed model
to deal with the single-row transformation of tree with all kinds
of structures. Hence, any node in the \( SP_9 \) can act as the root
node in the tree. To standardize the algorithm, the node with
the highest degree is chosen as the root node in the transformation.

C. TSM Model

The Tree Sequence Modeling (TSM) transforms the trees
into single-row networks. The transformation involves the
formation of zones, terminals and intervals. Formation of partitions
and insertion mechanisms are consisted in the
formation of zones. A tree is divided into a few partitions and
groups. There are two groups of nodes where each group has a
different arrangement for zones in insertion mechanisms.
Formation of zones optimizes the sequence of zones which
contributes to the optimal transformation result. The
arrangement of zones leads to the formation of shortest
intervals set with maximum number of zero energy nets in
order to obtain the optimal result.

After the formation of zones, the pins and the intervals are
formed using Formation of Terminals and Formation of Intervals [8]. The single-row routing problem is then solved
using the basic of simulated annealing [5] from earlier work called ESSR to obtain the result.

VI. EXPERIMENTAL RESULTS
STCGM has been applied on some connected graphs with different number of nodes. The results are simulated by a developed program for STCGM using Microsoft Visual C++ 6.0. The results for energy values (E), congestion (Q) and number of doglegs (D) are shown in Table I.

<table>
<thead>
<tr>
<th># nodes</th>
<th>d_{max}</th>
<th># d_{max}</th>
<th>E</th>
<th>Q</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>5</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>30</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>40</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>50</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>60</td>
<td>3</td>
<td>5</td>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

By having the same total number of nodes, the connected graphs with more communication links will give the spanning tree with lower value of D_{max}. The figures from Table I shows the results are not primarily affected by total number of nodes in G_n. In fact, the core factor is the maximum degree of node (D_{max}) in the spanning tree SP_n together with the total number of nodes with degree of D_{max} that exist in the SP_n. The SP_n with the same D_{max} may have the different results by having the different total number of nodes with degree of D_{max}. From the experimental results, the G_{50} and the G_{50} both have D_{max} of three. Since there is only one node that gives the D_{max} in G_{50} compared with three nodes in G_{50}, total energy for the G_{50} is then lower than the G_{50}. This shows the higher of D_{max} and the higher number of nodes with D_{max} in the spanning tree will lead to the higher value of total energy.

VII. CONCLUSION AND SUMMARY
In this research, a model which transforms the connected graph (G_n) into spanning tree (SP_n) before the single-row transformation into single-row network (S_n) is introduced, namely Spanning Tree of Connected Graph Modeling (STCGM). To obtain the optimal single-row transformation, the relationships between G_n to SP_n and SP_n to S_n are studied in order to find out the relationship from G_n to S_n. The connected graph G_n in this research has weight of one unit for all edges. From the study, the spanning tree with lower value of D_{max} which is the highest degree of node in SP_n was found to give the lower value of total energy for the complete transformation. Hence, STCGM is developed based on the finding.

STCGM consists Path-Growing Tree-Forming algorithm which is developed by applying Vertex-Prioritization to produce the SP_n with minimum value of D_{max} from the G_n. There are two parts in Path-Growing Tree-Forming algorithm. The Path-Growing produces the paths from the G_n and the Tree-Forming connects all these paths into a SP_n. Sometimes, Backward Tracing is needed in Path-Growing to form a walk in order to join the walk to the tree in Tree-Forming. After the SP_n is produced from the G_n, the S_n will be transformed into S_n using Tree Sequence Modeling (TSM). The single-row routing problem is then solved by ESSR [5] to obtain the result. The experimental results show STCGM can efficiently optimize the spanning tree transformation of connected graphs into single-row networks.

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