Coventry University
School of Engineering

PEPs

MSc Dissertation in

AUTOMOTIVE ENGINEERING

The Behavior of Vehicle during Cornering

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Date: 14 Sept 06
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DECLARATION

This dissertation is submitted as part fulfilment for the award of an MSC degree in

**Automotive Engineering**

The work is the result of my own investigations. All sections of the text and result which have been obtained from others workers/sources are fully referenced. I understand that cheating and plagiarism constitute a breach of University regulations and will be dealt with accordingly.

Signed: [Signature]

Date: 14th September 2006
Acknowledgement

I'd like to take this opportunity to express gratitude to my supervisors Mr. Gurmail Singh (Coventry University), Assc. Prof. Ir. Mustafar Ab Kadir and Mr. Wan Zailimi Wan Abdullah (KUTKM) who had provided useful information, guide, help and support throughout this study.

A lot of thanks also for Mr. Blundell (Coventry University) as an expert in Multibody System Analysis and Vehicle Dynamics who give ideas, help and suggestion in using ADAMS particularly using full vehicle model where his roll stiffness model is used as a based in this study.

I also would like to express my deep appreciation to my family members and all of my friends who contribute to this study by their own special ways. Only God can repay your help.
Abstract

This study described the use of multibody system analysis in studying the behaviour of the vehicle during cornering. Equivalent roll stiffness full vehicle model with Fiala tyre model is used to study understeer / oversteer characteristic of the vehicle during constant radius cornering test. Parameter affecting the vehicle behaviour is studied and discussed. An option in modelling full vehicle such as selection of steering system, steering input, suspension system representation and tyre model is also described.
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Chapter 1: Introduction

1.1 Simulation in engineering field

The use of simulation as a tool in analysing problem is well established especially in engineering field. With the growth of computing power, more and more researchers and industries have changed their experimental testing towards simulation. Back in the previous years where the computing power is much less than nowadays, product development depends on experimental testing and this is a costly operation and a time consuming process.

In modern days product development programme especially in automotive industry, time constraint and cost saving is crucial in maintaining a competitive product in market. In addition to that, changes in customer's taste and demand force manufacturers to rapidly change design variable to suit the market needs. The above factors made it necessary for manufacturer to find an alternative in handling product development problem and the solution to the problem is of course computer simulation.

With increasing of computing power of a factor of 100 or more over the last two decades (Blundell and Harty 2006), computer based analysis is rapidly growth. Starting with Finite Element Analysis (FEA) in the previous three decades and Multibody System Analysis (MBS) a decade after, both methods are widely used in engineering community and has become part or the vehicle design and development process.

Using simulation in product development is not only effective in term of cost saving and time consuming but also provides the ability to do parametric study towards certain design variable rapidly and without any risk. The use of simulation is therefore an important "weapon" in surviving the "war" in automotive industries.

In term of vehicle dynamic, Multibody System Analysis is the most well used method. It is used simulate the performance of a subsystems and full vehicle. Several commercial programmes are available for that purpose such as LMS.DADS and SIMPACK but the most well known software in this area is MSC.ADAMS.
ADAMS stands for Automatic Dynamic Analysis of Mechanical Systems and is well established in automotive companies and academic institution.

1.2 Cornering behaviour test: an overview

Vehicle cornering behaviour is gaining attention in measuring vehicle dynamics characteristic. This is proven by various report published related to that particular area (Katsuyama and Sakai 2005). Cornering behaviour is an important aspect in measuring a vehicle’s performance which is often equated as handling. “Handling” is a term refers to the responsiveness of a vehicle towards the driver’s input (Gillispie 1992) and shows the ease of control for a particular vehicle. Combination of a driver and vehicle is refers as “close loop” system where the driver will control the vehicle as desired. In this case, vehicle direction or position is observed by driver and the change in input (ie: yaw rate, brake) will be done to achieve the aimed motion. In contrast, “open loop” system does not involve interaction between drivers and vehicle. It concern only on the vehicle response towards a specific steering input. This method is used for characterising the vehicle and is defined as “directional response” behaviour (Gillispie 1992).

Open loop response is used to measure understeer gradient, a measure of performance under steady state condition. Society of Automotive Engineer (SAE) defined understeer gradient as “the quantity obtained by subtracting the Ackerman steer angle gradient from the ratio of the steering wheel angle gradient to the overall steering ratio” (Gillispie 1992). The experimental measurement of understeer gradient can be done using four methods; constant radius, constant speed, constant steer angle and constant throttle. Above all that are listed, only constant radius and constant speed method reflect the normal driving circumstances. Both methods are described as follows:
1.2.1 Constant radius method

Understeer gradient can be measured by operating the vehicle around a constant radius turn and observing steering angle versus lateral acceleration. As a start, vehicle is droved at a very low speed. At this speed lateral acceleration is almost negligible and steering wheel angle required in maintaining the turn is called Ackerman steer angle. Vehicle speed is then increased in steps (i.e. each speed will produce $\approx 0.1\text{g}$ increment in lateral acceleration) and the respective steer angle is recorded. Steering wheel angle (SWA) is then plotted against lateral acceleration as is shown in Figure 1.1.

![Diagram of steer angle vs lateral acceleration]

**Figure 1.1** Example measurement of understeer gradient by constant radius method (Source: Gillispie, 1992)

The slope of the steer angle curve is the understeer gradient. A positive slope indicate understeer, zero slope indicate neutral steer and negative slope indicate oversteer. Normally vehicles will behave either as understeer or oversteer for the entire operating range but some vehicle might be understeer at low speed and oversteer at high speed. Minimum radius of turn for this type of test is 30m (Gillispie 1992) but Blundell (2006) suggest a 33m instead.
1.2.2 Constant speed method

Understeer gradient can be measured at constant speed by varying steer angle. With this method, radius of turn will vary continuously and more data need to be collected to determine the gradient. Besides recording speed and steer angle, radius of turn at each case need also be collected. Figure 2.2 shows plotted of SWA vs lateral acceleration obtained using this method.

Figure 2.2 Example measurement of understeer gradient by constant speed method (Source: Gillispie 1992)

Straight line of constant slope shown is Ackerman steer angle gradient which indicate neutral steer. The instantaneous slope of the steer angle curve shows the behaviour of the vehicle; understeer or oversteer. If the slope is greater than the Ackerman steer angle gradient, the vehicle is understeer, if it is the same then the vehicle is neutrals teer, the rest indicate that the vehicle is oversteer.

This study will use constant radius test as it is more practicable (Blundell 2006) and easier to execute using computer simulation.
1.3 **Objective**

The objective of this case study is to investigate cornering behaviour of a vehicle by steady state cornering test using multibody system analysis software.

1.4 **Scope**

The scopes of the studies are:

1. using ADAMS full vehicle model, the model is not developed, instead using existing roll stiffness model developed by Blundell (2006)
2. focus on the behaviour of the vehicle during cornering by varying its rear roll stiffness
3. described several modelling technique of a full vehicle model and tire model available
4. discussed parameter affecting behaviour of the vehicle during cornering
Chapter 2: Introduction to ADAMS modelling

2.1 Basic principals in modelling

Before proceeding towards modelling the author feel that it is necessary to define what multibody system is. What is multibody system? Multibody system is a system comprised of multiple body connected by various means and can be analyzed using by application of Newtonian or Lagrangian method to formulate the equation of motion which may then be interrogated in a variety of ways – integrated through time, solved for eigen solution and so on (Blundell and Harty 2006).

First step in modelling multibody system using ADAMS is by preparing data set which will define the system being modelled. This includes description of rigid part, connecting joints, motion generators, forces and compliances.

For each rigid body in the system it is necessary to define the mass, centre of mass location, and mass moment of inertia. It is defined using command PART for example: PART/01, MASS=1.0, CM=0200, J=1,1,1. The statement shows that part #1 has a mass of 1 kg, centre of mass at point 0200 (which is defined later using coordinate system) and mass moment of inertia of 1 kg mm\(^2\) about x-axis, y-axis and z-axis.

Each body possesses a set of coordinate which is considered to move with the part during the simulation. The coordinate system which is define using command MARKER is used to define mass location, joint location and orientation, force location and directions. For example, in defining point 0200 command MARKER/0200, QP=100,100,100 is used. QP indicates the coordinate of point 0200 in x, y and z direction.

Body is connected between one to another using mechanical type joint to constraint it motion. Several commonly used joint are spherical, translational, universal, revolute and translational. Command JOINT is used together with I and J marker, where I part of a body is connected to the J part of the other body, for example JOINT/03, SPH, I=0403, J=0203 indicate joint #3 is a spherical and connected between part #4 and #2.

The last step is defining external and internal force element to the model. External force can be constant, time histories or functionally dependent on any state variable while
internal force element can be setup between two parts to represent spring, dampers, cables or rubber mount. External force can move with the part during simulation but for internal force it always acts along the line of sight where it connects the two parts.

2.2 Modelling example

To assist better understanding on modelling, static analysis of a double wishbone suspension is presented as an example. As a starting point, the brief schematic diagram of ADAMS’s model is shown in Figure 2.1 as follows:

![Diagram of double wishbone suspension](image)

**Figure 2.1** Schematic diagram of double wishbone suspension for ADAMS’s model

The suspension consists of seven parts (PART) which is mark with rectangular box. Each of the part is listed as below:

*PART 1: Body/ground*

*PART 2: Main upper control arm*

*PART 3: Main lower control arm*

*PART 4: Upper control arm inboard*

*PART 5: Lower control arm inboard*

*PART 6: Upper control arm outboard*

*PART 7: Lower control arm outboard*

*PART 8: Lower wishbone*

*PART 9: Wheel axis (hub)*
Part 1, 1 - Ground

Part 2, 2 - Upper arm

Part 3, 3 - Lower arm

Part 4, 4 - Steering knuckle

Part 5, 5 - Tie rod

Part 6, 6 - Damper (lower part)

Part 7, 7 - Damper (upper part)

Some of the part consists of several members and this member is connected with one another and with other part using joint (indicated using number in circle ie: (03)).

Part 2 - consist of 2 members, connected together and to part 4 using ball joint at #03 and to bushing at #01 and #02. The bush is connected to the ground.

Part 3 - consist of 2 members, connected together and to part 4 using ball joint at #06 and to bushing at #04 and #05. The bush is connected to the ground. The part is also connected to the part 6 using ball joint at #13

Part 4 - consist of 3 members, connected together at no #11, connected to part 2 using ball joint at #03, connected to part 3 using ball joint at #06 and to part 5 using ball joint at #07

Part 5 - consist of only 1 member, one end is connected to part 4 using ball joint at #07 and the other end is connected to the universal joint at #08. The universal joint
is connected to the ground.

Part 6 - consist of only 1 member, one end is connected to part 3 using ball joint at #13 and the other end is connected to part 7 using translational joint at #15.

Part 7 - consist of only 1 member, one end is connected to part 6 using translational joint at #15 and the other end is connected to ball joint at #14. The ball joint is connected to the ground.

Joint #13 and #14 is also connected using spring.

The connection between parts and joint is done using command JOINT and the connection is identified using I and J marker. Each of the part that need to have direction (ie: bushing, translational joint, cylindrical graphic and etc) must have the ZP marker where it defines the z-direction of the part relative to its global direction (parallel to the z-axis).

Table 1 show the coordinate of points in Figure 3 that need to be used in writing the ADAMS program. The 4-digit number (ie: 0100, 0200 etc) indicates the centre of mass for each part (ie: 0100 for part 1, 0200 for part 2 and etc)

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the load case for each program is inserted into the program using the command SFORCE. For example, for 3G bump case, the force 11180 N is inserted into the SFORCE in the z-direction whilst the force in x and y direction is kept zero:

SFORCE/1.l=0409,J=1,TRANS,ACTION,FUNC=0.0 !X-Force
SFORCE/2.l=0409,J=2,TRANS,ACTION,FUNC=0.0 !Y-Force
SFORCE/3,l=0409,J=3,TRANS,ACTION,FUNC=11180 !Z-Force

The full ADAMS programme for this case can be found in Appendix A.

2.3 Relationship between ADAMS simulation and traditional calculation method

Each of the symbol used in ADAMS language possess its own mathematical formulation where after the model is setup the mathematical equation formed is solved to obtain the solution. Each code and its formulation are described in details by Blundell (2006). This section describes the use of traditional calculation method to solve the same problem as for description of what’s happened behind the screen.

Before calculating the forces at the suspension’s component, the diagram of suspension is drawn which consist of notation of points and its coordinates. The free body diagram of the system is then formulated. (Please refer Appendix B)

For the body to be in static equilibrium, summation of force and moment acting on the body must be equal to zero, where in mathematical notation it is written as:

\[ \sum F_i = 0 \]
\[ \sum M_i = 0 \]

where F=force and M=moment
Hence,

For body 2,

Summation of force = 0

\[ \sum \{ F_2 \}_i = \{ 0 \}_i \]

\[ \{ F_{A21} \}_i + \{ F_{B21} \}_i + \{ F_{C24} \}_i = \{ 0 \}_i \]

writing the forces in matrix form for the component in x, y and z direction, we obtain:

\[
\begin{bmatrix}
F_{A21x} \\
F_{A21y} \\
F_{A21z}
\end{bmatrix} + 
\begin{bmatrix}
F_{B21x} \\
F_{B21y} \\
F_{B21z}
\end{bmatrix} + 
\begin{bmatrix}
F_{C24x} \\
F_{C24y} \\
F_{C24z}
\end{bmatrix} = 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

adding the forces together, we obtain 3 equations:

\[ F_{A21x} + F_{B21x} + F_{C24x} = 0 \] \(2.1\)
\[ F_{A21y} + F_{B21y} + F_{C24y} = 0 \] \(2.2\)
\[ F_{A21z} + F_{B21z} + F_{C24z} = 0 \] \(2.3\)

Summation of moment = 0

Taking the summation of moment about point C:

\[ \sum \{ M_{C2} \}_i = \{ 0 \}_i \]

\[ \{ R_{AC2} \}_i \times \{ F_{A21} \}_i + \{ R_{BC2} \}_i \times \{ F_{B21} \}_i = \{ 0 \}_i \]

the distance between point C and the forces can be calculated as follows:

\[
\begin{bmatrix}
2802 \\
397 \\
165
\end{bmatrix} - 
\begin{bmatrix}
2781 \\
696 \\
190
\end{bmatrix} = 
\begin{bmatrix}
21 \\
-299 \\
-25
\end{bmatrix}
\]

\[
\begin{bmatrix}
2500 \\
470 \\
117
\end{bmatrix} - 
\begin{bmatrix}
2781 \\
696 \\
190
\end{bmatrix} = 
\begin{bmatrix}
-281 \\
-226 \\
-73
\end{bmatrix}
\]
Hence, the cross product can be written in the matrix form:

\[
\begin{bmatrix}
0 & 25 & -299 \\
-25 & 0 & -21 \\
299 & 21 & 0
\end{bmatrix}
\begin{bmatrix}
F_{A21x} \\
F_{A21y} \\
F_{A21z}
\end{bmatrix}
+ \begin{bmatrix}
0 & 73 & -226 \\
-73 & 0 & 281 \\
226 & -281 & 0
\end{bmatrix}
\begin{bmatrix}
F_{B21x} \\
F_{B21y} \\
F_{B21z}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

by multiplying and adding the matrix, we can obtain another 3 equations which are:

\[25F_{A21y} - 299F_{A21z} + 73F_{B21y} - 226F_{B21z} = 0\] ..................................................2.4

\[-25F_{A21x} - 21F_{A21z} - 73F_{B21x} + 281F_{B21z} = 0\] ..................................................2.5

\[299F_{A21x} + 21F_{A21y} + 226F_{B21x} - 281F_{B21y} = 0\] ..................................................2.6

applying the same principal for body 3 and body 4, we can obtain:

For body 3,

Summation of force = 0

\[\sum \{F_3\}_i = \{0\}_i\]

\[\{F_{D31}\}_i + \{F_{E31}\}_i + \{F_{F3}\}_i + \{F_{K36}\}_i = \{0\}_i\]

to reduce the number of unknown, the scale factor is applied to the force where the line of action in known (2 force members) and obtain:

\[\{F_{K36}\}_i = f_{S1} \{R_{KL}\}_i\]

where \[\{R_{KL}\} = R_K - R_L\]

\[R_K - R_L:
\begin{bmatrix}
2754.5 \\
631 \\
-25.5
\end{bmatrix}
- \begin{bmatrix}
2706.5 \\
533 \\
200.5
\end{bmatrix}
= \begin{bmatrix}
48 \\
98 \\
-226
\end{bmatrix}\]

therefore,

\[\begin{bmatrix}
F_{K36x} \\
F_{K36y} \\
F_{K36z}
\end{bmatrix}
= f_{S1}
\begin{bmatrix}
48 \\
98 \\
-226
\end{bmatrix}\]
hence, the general equation can be written in the matrix form as follows:

\[
\begin{bmatrix}
F_{D31x} \\
F_{D31y} \\
F_{D31z}
\end{bmatrix} + \begin{bmatrix}
F_{E31x} \\
F_{E31y} \\
F_{E31z}
\end{bmatrix} + \begin{bmatrix}
F_{F34x} \\
F_{F34y} \\
F_{F34z}
\end{bmatrix} + f_{S1} \begin{bmatrix}
48 \\
98 \\
-226
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

adding the forces together, 3 more equations can be obtained:

\[F_{D31x} + F_{E31x} + F_{F34x} + 48f_{S1} = 0 \] .............................2.7
\[F_{D31y} + F_{E31y} + F_{F34y} + 98f_{S1} = 0 \] .............................2.8
\[F_{D31z} + F_{E31z} + F_{F34z} - 226f_{S1} = 0 \] .............................2.9

Summation of moment at point F = 0

\[\sum \{M_{F3}\}_1 = \{0\}_1\]

\[\{R_{DF}\}_1 \times \{F_{D31}\}_1 + \{R_{EF}\}_1 \times \{F_{E31}\}_1 + \{R_{KF}\}_1 \times \{F_{K36}\}_1 = \{0\}_1\]

\[R_D - R_F:\]
\[
\begin{bmatrix}
2803 \\
314 \\
-24
\end{bmatrix} - \begin{bmatrix}
2803 \\
730 \\
-26
\end{bmatrix} = \begin{bmatrix}
0 \\
-416 \\
\phantom{0}2
\end{bmatrix}
\]

\[R_E - R_F:\]
\[
\begin{bmatrix}
2418 \\
357 \\
-24
\end{bmatrix} - \begin{bmatrix}
2803 \\
730 \\
-26
\end{bmatrix} = \begin{bmatrix}
-385 \\
-373 \\
\phantom{0}2
\end{bmatrix}
\]

\[R_K - R_F:\]
\[
\begin{bmatrix}
2754.5 \\
631 \\
-25.5
\end{bmatrix} - \begin{bmatrix}
2803 \\
730 \\
-26
\end{bmatrix} = \begin{bmatrix}
-48.5 \\
-99 \\
\phantom{0}0.5
\end{bmatrix}
\]

hence;

\[
\begin{bmatrix}
0 & -2 & -416 \\
2 & 0 & 0 \\
416 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & -2 & -373 \\
2 & 0 & 385 \\
373 & -385 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0.5 & -99 \\
0.5 & 0 & 48.5 \\
99 & -48.5 & 0
\end{bmatrix} \begin{bmatrix}
F_{D31x} \\
F_{D31y} \\
F_{D31z}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0.5 & -99 \\
0.5 & 0 & 48.5 \\
99 & -48.5 & 0
\end{bmatrix} \begin{bmatrix}
48f_{S1} \\
98f_{S1} \\
-226f_{S1}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]