ASPERICAL WEIGHT TEST FOR MIXED MONOID PRESENTATION

TAY CHOO CHUAN and ABD GHAFUR BIN AHMAD
Pusat Pengajian Sains Matematik, Fakulti Sains dan Teknologi
Universiti Kebangsaan Malaysia, Bangi, 43600, Malaysia

Abstract: This paper constructs generalized left and right graph for mixed monoid presentation. Then the asphericality test is given.

1. Introduction

A mixed monoid presentation is a quadruple
\[ M = \{ a, t \} \times \{ s, t \} \]
where \( a \) and \( s \) are generators and relators respectively such that the group presentation
\[ P = \langle a \mid s \rangle \]
defines a group \( G \). Thus \( s \) is a set of cyclically reduced words on \( a \). The set \( t \) is a finite positive alphabet such that for each element \( R \in F \) can be represented by
\[ (R: R = R, \epsilon = \pm 1) \]
where \( R = R \) are positive reduced words on \( a \cup t \) containing at least element of \( t \) in the form of
\[ h_i h_2 \ldots h_m = \tau_1^i \tau_2^j \ldots \tau_n^k \ (h_i, h_j, \ldots, h_m) \in G, \tau_i \in \tau t \]

With the mixed monoid presentation \( M \), we associate a graph \( \Gamma = \gamma M \) as follows. The graph \( \Gamma \) has vertex set of all words on \( a \cup t \), and the edge set consists of all atomic relative pictures.

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\[ E = (U, R_e \rightarrow R_{-e}, V) \]

such that \( U = U_1 t_h U_2 t_h \ldots U_{m-1} t_h U_m \) and \( V = V_1 t_h V_2 t_h \ldots V_{m-1} t_h V_m \), where \( t_{w_0} \in t \), while \( U \) and \( V \) are words on \( a \).

In \( E \), \( R_e \rightarrow R_{-e} \) has the form \( s_{1} t_{1} s_{1} \ldots t_{m} s_{m} \rightarrow h_{1} t_{1} h_{1} \ldots t_{m} h_{m} \). This atomic relative picture is depicted below.

The initial, terminal and inverse functions are respectively given by

\[ \iota(E) : UR_e V, \tau(E) : = UR_e V, E^{-1} = (U, R_{-e} \rightarrow R_e, V) \]

A paths in \( \Gamma \) is called relative picture over \( \mathcal{M} \) and a closed path is called spherical relative picture over \( \mathcal{M} \). If \( \Gamma \) does not have any reduced closed path, then we say that \( \mathcal{M} \) is a spherical.

Normally if there exist a reduced closed path in \( \Gamma \), or in term of picture there exist a reduced spherical picture over \( \mathcal{M} \). Thus our definition of asphericity is as defined by Pride [3] and not as defined by Guba and Sapir [2].

Kilgour [5] introduced a cycle free test in order to determine the asphericity of any given mixed monoid presentation. Several mixed monoid presentations have been determined by Sim [6]. Unfortunately, this test is too general and can not be applicable in various type of mixed monoid presentations.
By generalizing Kilgour’s cycle test [3] and reorganizing Bogley and Pride relative test [1], we introduce a new test generalized weight test in order to determine the sphericity of any given mixed monoid presentation.

2. Generalized Left and Right Graph

Let $\mathcal{M} = [a, [x, r]]$ be a mixed monoid presentation. If $R_{l}(R_{e} = R_{w}) \in r$ then let $^{R^{l}_{l}(R^{l}_{e} = R^{l}_{w})}$ where $^{R^{l}_{l}}$ and $^{R^{l}_{e}}$ are all the possible relative permutation of $R_{l}$ and $R_{e}$ respectively.

For example:

If $R_{l} : (a, b, c, t_{1}, t_{2}, t_{3}, t_{4}, t_{5})$ and $t_{i}, t_{i}' \in t_{1}, a, b, c, a', b' \in G$ then we have,

$^{1}R^{l}_{l}(a, b, c, t_{1}) = i_{a}'i_{b}'$

$^{1}R^{l}_{l}(a, b, c, t_{2}) = i_{t_{1}}i_{b}'$

$^{1}R^{l}_{l}(a, b, c, t_{3}) = i_{c}'i_{b}i_{t_{1}}$

$^{1}R^{l}_{l}(a, b, c, t_{4}) = i_{t_{2}}i_{b}'i_{t_{1}}$

$^{1}R^{l}_{l}(a, b, c, t_{5}) = i_{t_{3}}i_{b}'i_{t_{1}}$

For each $^{1}R^{l}_{l}(a, b, c, t_{i}, t_{i}'), t_{1}, a, b, c, a', b' \in G$, let

$^{(a, b, c, t_{i}, t_{i}' )}_{l} = i_{a}'i_{b}'i_{c}'i_{t_{i}}i_{t_{i}'}$

$^{(a, b, c, t_{i}, t_{i}' )}_{l} = i_{t_{i}}i_{b}'i_{t_{i}'}i_{t_{i}'}$

Thus we obtain a graph called the generalized left graph of $\mathcal{M}$ denoted by $\text{GLG}(\mathcal{M})$. Similarly we define,

$^{R^{l}_{l}(R^{l}_{e} = R^{l}_{w})}_{l} = i_{a}'i_{b}'i_{c}'i_{t_{i}}i_{t_{i}'}$

$^{R^{l}_{l}(R^{l}_{e} = R^{l}_{w})}_{l} = i_{t_{i}}i_{b}'i_{t_{i}'}i_{t_{i}'}$

$^{H^{l}_{l}(R^{l}_{w})}_{l} = h_{l}^{i_{a}i_{b}i_{c}i_{t_{i}}}$
Lemma 1 Let $c_j, \ldots, c_{j_n}$ be the sequence of corners encountered in an anti-clockwise traverse of an inner region $\Sigma$ of a picture $\mathcal{P}$ over $\mathcal{M}$, then the sequence of edges $W(c_j), \ldots, W(c_{j_n})$ is a cycle in GLG($\mathcal{M}$) or GRG($\mathcal{M}$).

Proof If an arc $\alpha$ meets the corners $c_j$ and $c_{j_n}$ in the boundary of $\Sigma$, then it must joint the head of $c_{j_n}$ to the tail of $c_j$ (or vice versa). From this it follows that $\tau(W(c_j))=\tau(W(c_{j_n}))$ (the label on $\alpha$).

3. Weight Test

A weight function $\theta$ on GLG($\mathcal{M}$) and GRG($\mathcal{M}$) is a real valued function on the set of edges of GLG($\mathcal{M}$) and GRG($\mathcal{M}$) which satisfies

\[ \theta(\phi_k(e^{-1}_{i_t})) = \theta(\phi_k(e_{i_t}^{e^{-1}})) \quad \text{and} \quad \theta(\phi_k(e_{i_t}^{e^{-1}})) = \theta(\phi_k(e_{i_t})). \]

The weight of a path is the sum of the weights of the constituent edges.

A weight function $\theta$ on GLG($\mathcal{M}$) and GRG($\mathcal{M}$) is said to be aspherical if the following two conditions are satisfied.

(3.1) Let $R \in \mathcal{P}$, say $R=x_1^{e_n}h_1 \ldots x_n^{e_n}$.

Then $\sum_{i=1}^{n}(1-\theta(x_i^{e_i}h_i \ldots x_n^{e_n}) \geq 2$.

(3.2) Each admissible cycle in GLG($\mathcal{M}$) and GRG($\mathcal{M}$) has weight at least 2. An aspherical weight function on GLG($\mathcal{M}$) is aspherical if each edge of GLG($\mathcal{M}$) and GRG($\mathcal{M}$) has non-negative weight.

Theorem 2. If GLG($\mathcal{M}$) or GRG($\mathcal{M}$) admits an aspherical weight function, then $\mathcal{M}$ is aspherical.

Proof: Suppose that $\mathcal{P}$ is a reduced connected spherical picture over $\mathcal{M}$, and that GLG($\mathcal{M}$) and GRG($\mathcal{M}$) admit an aspherical weight function $\theta$.

Shrink each disc of $\mathcal{P}$ to a vertex, and identify $\partial \mathcal{P}$ to a point to obtain a tessellation $T$ of the sphere. Let $n_v, n_e, n_f$ be the number of vertices, edges, faces of $T$. Thus $n_f$ is equal to the number of discs of $\mathcal{P}$, $n_v$ is equal to the number of regions. Denote the set of corners of $\mathcal{P}$ by $C$. Clearly, $2n_v = \sum_{r \in C} 1$. 


Summing over all vertices of $T$, the condition (3.1) implies $2n_1 \leq \sum_{v \in \mathcal{C}} (1 - \theta(W(v)))$.

Now, each inner region of $P$ supports an admissible cycle in $GLG(\mathcal{M})$ and $GRG(\mathcal{M})$. Moreover, since $P$ is aspherical, the outer annular region also supports an admissible cycle in $GLG(\mathcal{M})$ and $GRG(\mathcal{M})$. Thus, summing over all faces of $T$, the condition (3.2) implies $2n_1 \leq \sum_{v \in \mathcal{C}} \theta(W(v))$.

We now obtain the following contradiction:

$$2 = n_\alpha - n_\gamma + n_\Omega = \frac{1}{2} \sum_{v \in \mathcal{C}} [(1 - \theta(W(v))) - 1 + \theta(W(v))] = 0$$

Therefore, with the method of contradiction, we can conclude that if $GLG(\mathcal{M})$ or $GRG(\mathcal{M})$ admits an aspherical weight function, then $\mathcal{M}$ is aspherical.

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