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COMPARISON OF HARMONIC BALANCE AND MULTIPLE SCALE METHOD ON DEGREE OF NONLINEARITY FOR DUFFING OSCILLATOR

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ABSTRACT

The well-known approximate solution of harmonic balance is compared with one of the techniques of perturbation, the multiple scale method. Both chosen analytical method are compared in terms of degree of nonlinearity of a hardening base excitation system. The assumed solution in harmonic balance is taken to single mode while multiple scale solution is assumed to have the solution of a first order expansion. Both methods are solved to attain the frequency response equation and the response of both curves are plotted and compared to show the differences in each method with the same value of nonlinearity. In the chosen parameter of nonlinearity, $\alpha = 0.03$ and $\alpha = 3.5$, harmonic balance method appears to tilt at a higher degree than that of multiple scale method thus showing higher accuracy in terms of nonlinearity from the comparison.

Keywords: hardening, harmonic balance method, multiple scale method.

INTRODUCTION

Energy harvesting from ambient vibrations has been attracting researchers in recent few decades with its promising ability to power applications included and hostile environment. The energy harvesting device is modelled as single-degree-of-freedom mass-spring-damper system either mass-excited or base-excited conventionally. In such model system, the energy harvested is equivalent to the energy dissipated and the performance is optimized when the natural frequency of the device is tuned to match the ambient frequency (Williams and Yates, 1996). This limit the performance of the system as the ambient frequency may not be tonal with time and a slight mistune would depreciate the performance of the device significantly. In order to cater this restriction, active tuning and passive tuning mechanism is implemented so as to ensure that the response of the device is insensitive to the change in frequency through altering the natural frequency of the device to the ambient frequency (Zhu *et al*., 2010; Tang *et al*., 2010). Another alternative that researchers are looking into is in altering the stiffness of the system, or in other words broadening the bandwidth of the system through the inclusion of nonlinearity (Ramlan *et al*., 2010). The stiffness nonlinearity is introduced by replacing a linear spring with a hardening nonlinear spring since it is capable to bend the response to the right and decrease the system's sensitivity in terms of frequency variation. The preceding alternative appears to be promising as the spring itself already acts as a tuning mechanism without the need of any additional tuning mechanism.

Regardless of the proven bandwidth widening ability presented by the hardening nonlinear spring, the closed form performance characteristics in terms of coupling strength has been presumed to converge to the linear system with the assumption that the degree of nonlinearity of the system is weak. A comparison of an analytical analysis approach is presented to quantify the degree of nonlinearity of a hardening nonlinear spring system. This would serve to determine the upper bound limits of the performance of the nonlinearly coupled device.

LITERATURE REVIEW

Base excitation system

Energy harvesting device is commonly modelled as a base excited single-degree-of-freedom (SDOF) system as shown in Figure-1 where *m* represents mass of the system, *k* represents stiffness of the system, *c* represents damping of the system, *x* represents sinusoidal force applied to mass and *y* represents sinusoidal base force.

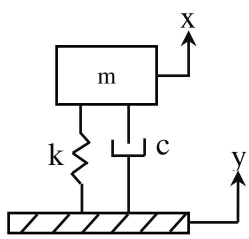

Figure-1. SDOF base excitation system.

Over the time, the study of a typical linear energy harvesting device has been carried out analytically and experimentally (Roundy, 2005; Stephen, 2006; Chitta *et al*., 2006; Niell and Alex, 2011; Evan *et al*., 2012) presenting the limit of a linear resonant generator, that is the performance of the system is at optimum when the natural frequency of the device is matched with ambient frequency but worsen when vice versa as shown in Figure-2.

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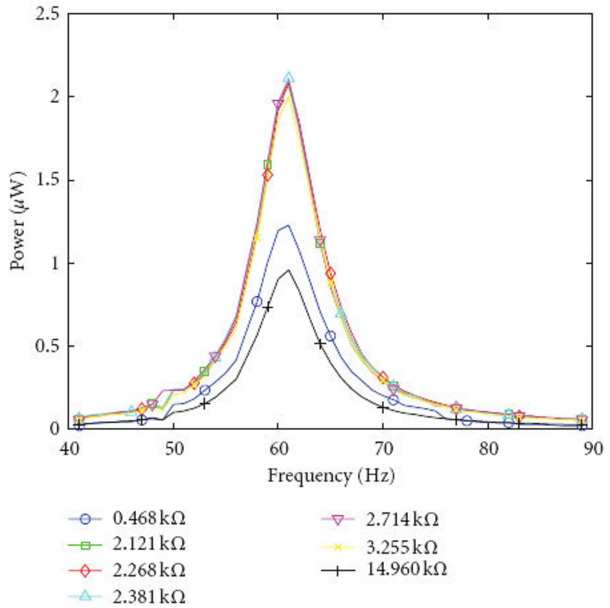

Figure-2. Frequency response curve of a linear SDOF base excitation system of an electromagnetic transducer with variation of resistive load [source: Evan *et al.*, 2012].

The introduction of a hardening nonlinear stiffness to overcome the limitation prompted by the linear system has been studied analytically and experimentally too by (Brennan et al., 2008; Mann and Sims, 2009; Barton *et al.*, 2010; Ramlan *et al.*, 2010). Figure-3 shows the proven widening bandwidth of a nonlinear system with a hardening spring.

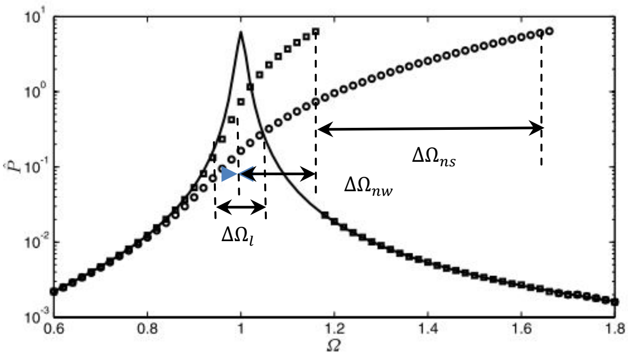

Figure-3. Frequency response curve of a linear and hardening nonlinear base excitation system. Linear spring (solid line); linear bandwidth, $\Delta\Omega_l$, hardening spring with a weak nonlinearity (square); narrow bandwidth, $\Delta\Omega_{nw}$ and hardening spring with a strong nonlinearity (circle); widen bandwidth, $\Delta\Omega_{ns}$ [source: Ramlan *et al.*, 2010].

Nonlinear oscillations - analytical approach

Nonlinear oscillations are studied analytically through the approach of approximate solution and perturbation techniques. These analytical approach are discussed in (Nayfeh, 1979 and Nayfeh, 1981) explaining in details all the available methods.

The approximate solution method is known as the harmonic balance method where one would solve the equation of motion of the system by expressing the periodic solution to be in the form of $u = A\cos T$ (A is the steady state response amplitude) and this method is said to be not restricted to weakly nonlinear problems only by (Hamdan and Burton, 1992).

A few perturbation techniques are available for solving the nonlinear oscillations and that involved the straightforward expansion, the Lindstedt-Poincare method, the multiple scales method and the averaging method. All these perturbation techniques have been applied by (Burton and Rahman, 1986; Cheung *et al.*, 1991; Xu and Cheung, 1994) in analysing nonlinear oscillators system.

METHODOLOGY

The approximate solution of harmonic balance method and the multiple scales method from the perturbation techniques are chosen to perform the analytical analysis of a base excited hardening nonlinear system. Harmonic balance method is assumed to have the solution in terms of single mode as written in equation (4) while multiple scales method is solved up to first order expansion. Both method were expressed in the amplitude - frequency relationship for comparison.

The equation of motion of a base excitation system is written as

$$m\ddot{s} + c\dot{s} + k_1 s + k_3 s^3 = -m\ddot{y} \qquad (1)$$

where $s = x - y$ is the relative displacement between the seismic mass, x and the housing, y, c is the damping coefficient, k_1 is linear stiffness and k_3 is nonlinear stiffness.

Equation (1) when expressed in the form of a non-dimensionless hardening nonlinear Duffing oscillator becomes

$$\ddot{u} + 2\zeta\dot{u} + u + \alpha u^3 = P\cos\Omega t \qquad (2)$$

where $2\zeta = \frac{c}{m\omega_n}$, $\alpha = \frac{k_3 Y^2}{k_1}$, $P = \frac{w^2}{\omega_n^2}$.
ζ –'damping of the system, α – nonlinearity and P – excitation given to the system.

Harmonic balance method

A new independent variable $T = \Omega t$ is introduced to (2) to gain

$$\Omega^2\ddot{u} + \Omega 2\zeta\dot{u} + u + \alpha u^3 = P\cos(T + \phi) \qquad (3)$$

where dots denote T derivatives and ϕ is the phase angle between the excitation and response.

Equation (3) is assumed to have the solution as follow:

$$u(T) = A\cos T \qquad (4)$$

Expressing equation (4) into equation (3) yield

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$$-\Omega^2 A\cos T + A\cos T + \frac{3}{4}\alpha A^3\cos T - 2\zeta\Omega A\sin T = P\cos T\cos\phi - P\sin T\sin\phi \quad (5)$$

Equating coefficients of (5) gain

$$(1-\Omega^2)A + \frac{3}{4}\alpha A^3 = P\cos\phi \quad (6)$$

$$-2\zeta\Omega A = P\sin\phi \quad (7)$$

Squaring and adding equation (5) and (6) gives

$$A^2\Omega^4 + (A^2 - 2A^2 - \frac{3}{2}\alpha A^4 + (2\zeta A)^2)\Omega^2 + \frac{3}{2}\alpha A^4 + \left((\frac{3}{4}\alpha A^3)^2 - P^2\right) = 0 \quad (8)$$

Solving for Ω^2 as a function of A yield

$$\Omega^2 = 1 + \frac{3}{4}\alpha A^2 + 2\zeta^2 \pm \sqrt{(\frac{P}{A})^2 - 3\alpha\zeta^2 A^2} \quad (9)$$

Multiple scales method

Consider equation (2) where ε is introduced as a book keeping parameter and expressed as follow:

$$\ddot{u} + 2\varepsilon\zeta\dot{u} + u + \varepsilon\alpha u^3 = \varepsilon P\cos\Omega t \quad (10)$$

The assumed solution of (10) is written as a first order expansion

$$u(t;\varepsilon) = u_0(t) + \varepsilon u_1(t) + .. \quad (11)$$

Where $t = t(T_0, T_1, ..)$ and $T_n = \varepsilon^n t$ so that the derivatives are transformed according to

$$\frac{d}{dt} = D_0 + \varepsilon D_1 \quad (12a)$$

$$\frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 \quad (12b)$$

Substituting equation (11) and (12) into equation (10) and sorting into orders of ε gives the following two linear equations

$$\varepsilon^0 : D_0^2 u_0 + u_0 = 0 \quad (13a)$$

$$\varepsilon^1 : D_0^2 u_1 + u_1 = -2D_0 D_1 u_0 - 2\zeta D_0 u_0 - \alpha u_0^3 + \varepsilon P\cos\Omega t \quad (13b)$$

where the terms ofε^2 and higher have been neglected as defined by equation (11). Solution of equation (13a) is assumed to be

$$u_0 = A(T_1)e^{iT_0} + \bar{A}(T_1)e^{-iT_0} \quad (14)$$

where$A(T_1)$ and $\bar{A}(T_1)$ are complex conjugates that are functions of the higher time scale. Substituting equation (14) into equation (13b) brings

$$D_0^2 u_1 + u_1 = -2i(A' + \zeta A)e^{iT_0} - (3\alpha A^2\bar{A})e^{iT_0} + \varepsilon P\cos\Omega t + cc \quad (15)$$

where the prime represents derivative with respect to T_1 and cc represents the complex conjugates of the right hand side terms. A more accurate frequency response near resonance of such a system is of high interest and therefore the substitution of $\Omega = 1 + \varepsilon\sigma$ is introduced into (15) to describe the nearness of Ω to resonance. The term σ is known as a detuning parameter and is use to describe the nearness of Ω to resonance. After substitution, equation (15) yield

$$D_0^2 u_1 + u_1 = -2i(A' + \zeta A)e^{iT_0} - (3\alpha A^2\bar{A})e^{iT_0} + Pe^{iT_0}e^{i\sigma T_1} + cc \quad (16)$$

The coefficients of e^{iT_0} in equation (16) are known as secular terms and are set to zero to eliminate the unbounded solution cause by these terms. Setting them to zero bring

$$-2i(A' + \zeta A) - (3\alpha A^2\bar{A}) + Pe^{\sigma T_1} = 0 \quad (17)$$

Polar form of $A = \frac{1}{2}a(T_1)e^{i\phi T_1}$is introduced to (17) where a and ϕ are functions of slow time scales T_1 and upon simplification yield

$$-i(a' + ai\phi + \zeta a) - \frac{3}{8}\alpha a^3 + Pe^{i(\sigma T_1 - \phi)} = 0 \quad (18)$$

Substituting $\gamma = \sigma T_1 - \phi$ into equation (18) and separating the resulting equation into real and imaginary part gives

$$a\gamma' = \sigma a - \frac{3}{8}\alpha a^3 + P\cos\gamma \quad (19a)$$

$$a' = -\zeta a + P\sin\gamma \quad (19b)$$

By setting $\gamma' = 0$ and $a' = 0$, the steady state solutions of equation (19a) and (19b) can be found. Squaring and adding these relationships results in the following equation

$$(\sigma a)^2 - \frac{3}{4}\sigma\alpha a^4 + \frac{9}{64}\alpha^2 a^6 + (\zeta a)^2 = P^2 \quad (20)$$

Substituting $\Omega = 1 + \varepsilon\sigma$ where $\sigma = \frac{\Omega - 1}{\varepsilon}$ into equation (20) yield

$$a^2\Omega^2 + \left(-2a^2 - \frac{3}{4}\varepsilon\alpha a^4\right)\Omega + a^2 + \frac{3}{4}\varepsilon\alpha a^4 + \frac{9}{64}\varepsilon^2\alpha^2 a^6 + (\zeta\varepsilon a)^2 - (\varepsilon P)^2 = 0 \quad (21)$$

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Solving Ω as a function of a results in the following equation

$$\Omega = 1 + \frac{3}{8}\varepsilon\alpha a^2 \pm \varepsilon\sqrt{(\frac{P}{a})^2 - \zeta^2} \quad (22)$$

RESULTS AND DISCUSSIONS

Frequency response curve

The frequency-amplitude relation obtained from the harmonic balance method and multiple scales method which is the equation (9) and equation (22) are presented in Figure-4 and Figure-5. Equation (9) and (22) are plotted with the following constant parameters, $\zeta = 0.07$, $P = 0.8$ and $\varepsilon = 0.1$ while the value of αvaried between Figure 4 and Figure 5 so as to compare the degree of nonlinearity that can be presented by both methods.

In Figure-4, graph of amplitude versus frequency is presented for both harmonic balance and multiple scales method. As can be seen from the graph, the frequency response curve attain from harmonic balance method tends to tilt to the right at a greater degree compared to the curve obtained from multiple scales method that only slant slightly to the right. This means that at the value of $\alpha = 0.03$, harmonic balance method can already shows the hardening nonlinear of the system obviously than multiple scales method that appears almost like a linear frequency response.

Graph of Figure-5 is plotted with the value of $\alpha = 3.5$ in order to illustrate the results obtained by both method in a higher degree of nonlinearity. The characteristics of the graph appears to be similar to Figure-4 where the frequency curve attained from harmonic balance method apt to tilt to the right at a higher degree than that of multiple scales frequency response curve. Though the frequency curve of the multiple scales method also tilted to the right in the graph in a noticeably way, the harmonic balance frequency response curve tilted more than the multiple scales one representing its capability to demonstrate the nonlinearity of the system. The results presented appears to be in line with the theory addressed by (Hamdan and Burton, 1992), which is to say that the harmonic balance method is not restricted to weakly nonlinear problems only.

The results obtained indicate that the harmonic balance method is able to illustrate the nonlinearity of the system in a much sensitive level compared to the multiple scales method as the tendency of bending to the right represents the higher hardening nonlinear the system is possessing. The reason for such a result may be explained in such a way that the harmonic balance method equation is solved in Ω^2(refer equation (9)) which is consistent with the governing differential equation that is of second order and thus offers a more accurate results. Multiple scales method on the other hand is solved in Ω (refer equation (22)) in a way that is not consistent with is second order governing differential equation and ended up presenting a much lower accuracy results than the harmonic balance method.

A weak nonlinear system as presumed in Figure-4 by multiple scales method, harmonic balance method illustrate that the system has actually possess enough nonlinearity and the consequences of assuming the coupling strength of the system to converge to the linear system may not be favourable at all.

CONCLUSIONS

Harmonic balance method appears to be capable of determining the degree of nonlinearity of a hardening system in a much accurate way than the multiple scales method. Regardless of a weakly nonlinear hardening system or a strong nonlinear hardening system, harmonic balance method just proves its way as shown in the results of Figure 4 and Figure 5. This is thus beneficial in determining the upper bound limit of a nonlinear hardening system. Once the upper bound limit is determined, the coupling strength of a weak nonlinear system would not be preferable to be assumed to converge to linear system any longer.

In future work, the multiple scales method may be worked out to be in order of Ω^2 and then compared with harmonic balance method again to check on the accuracy of the results. Furthermore, the closed form performance of a weak nonlinear hardening system in terms of coupling strength can be studied by taking the weak nonlinear into account instead of assuming to converge with the linear one. Also, an energy harvesting device employing the hardening nonlinear characteristic could be fabricated to gather experimental results and then use to compare with the obtained analytical results.

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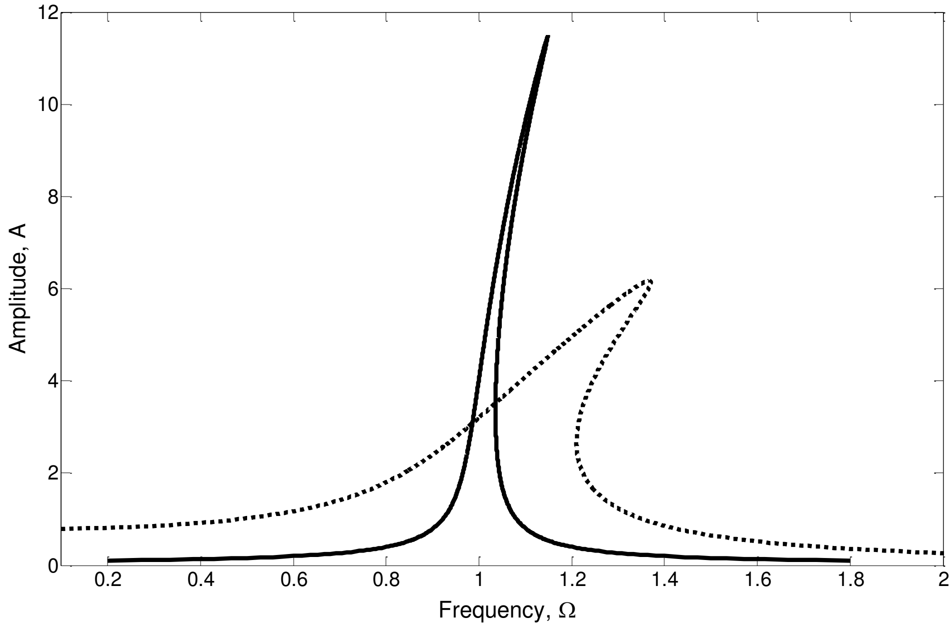

Figure-4. Graph of amplitude, A versus frequency, Ω (non-dimensional) plottedfor$\alpha = 0.03$,$\zeta = 0.07$, $P = 0.8$ and $\varepsilon = 0.1$ for harmonic balance method (dashed) and multiple scales method (solid)

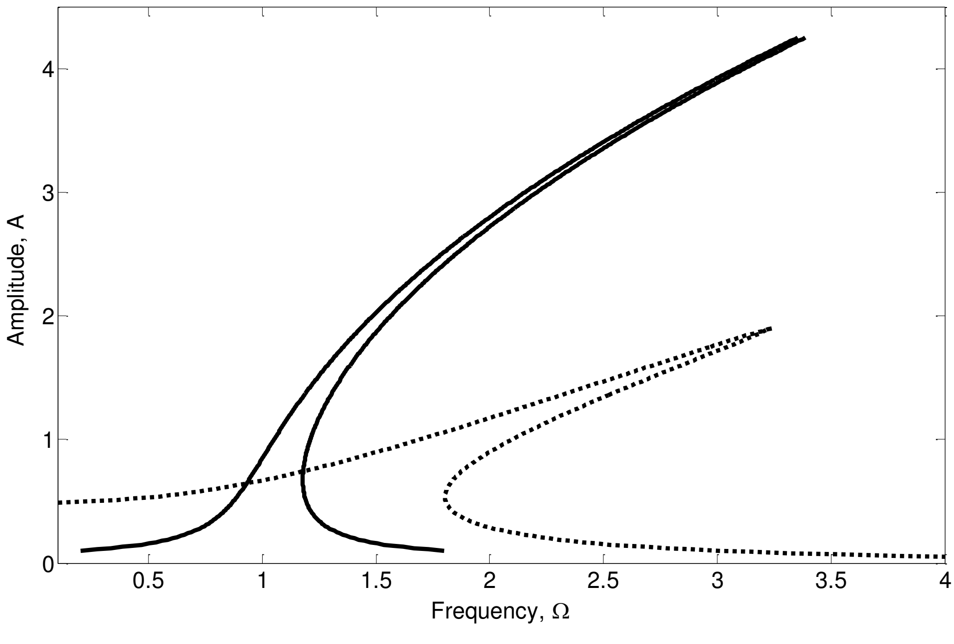

Figure-5. Graph of amplitude, A versus frequency, Ω(non-dimensional) plotted for $\alpha = 3.5$, $\zeta = 0.07$, $P = 0.8$ and $\varepsilon = 0.1$ for harmonic balance method (dashed) and multiple scales method (solid).

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