

Linearization - Advantages and Shortcomings Toward Control System Design

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Abstract – *This brief manuscript discusses the necessity to linearize nonlinear systems. Thorough review on nonlinear phenomena in dynamical and numerical system is presented. The methodology to linearize nonlinear system in Jacobian approach is shown in didactic manner. Numerical and dynamical example of nonlinear system is provided to enhance understanding. Afterward, the comparison between both linearized and non-linearized system is literally discussed. The outcomes concluded that linearization process is a linear approximation of a nonlinear system that is only valid in a small region around an operating point.*

Keywords: *Jacobian, Linear system, nonlinear system, equilibrium point, Linearization*

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I. Introduction

Many physical systems are nonlinear in nature. Engineers involve in multifarious technologies ranging from renewable energy [1]–[2], electro-mechanical systems [3], manufacturing industries [4]–[5], robotics [6]–[7], aerospace [8], maritime [9] and automotive sectors [10] are dealing with nonlinear phenomena. In μ -nano systems, nonlinearity comes from hysteresis phenomena, friction, as well as discontinuous behavior [11]. Robotic systems hold nonlinearity due to Sine or Cosine functions. Whereas, study in wind turbine technology shows that the power coefficient characteristic of typical wind turbine system behaves as a nonlinear function [12]–[13]. Likewise, backlash appears in rotational geared mechanical system introduces nonlinearity phenomena [14]. If the backlash is accounted in the system dynamics, the system model behaves as nonlinear function which is difficult to control. Practically, the backlash is known to be insignificant to the system dynamics and therefore being neglected by the control designer.

Nonlinear systems do not fulfill superposition principle as linear systems do. Nonlinear systems absorb nonlinear phenomena such as chaos, limit cycle, saturation, finite escape time. Nonlinear systems have

multiple isolated equilibrium points, or sometimes infinite and even not exist. If the nonlinearity is too large and significant to the system dynamics, the performance of the system may be deteriorated when the system is controlled by linear controller derived through linearized model [15]. The controllability and observability of nonlinear system is also hard to prove. The stability can be proved by using high level mathematical manipulation such as Lyapunov [16]–[18] or Popov [19]–[20] through Nyquist. However, frequency analysis for nonlinear system is almost impossible in order to facilitate Nyquist criteria. Thus, solving nonlinear systems requires advance control techniques. The presence of exogenous disturbances and uncertainties in the nonlinear systems dynamic is sometimes inevitable, and give catastrophic effect to the stability and robustness of closed loop systems.

Nonlinear phenomena sometimes necessary as adopted by oscillator and cyclic systems. Some strange behaviors of nonlinear systems can be observed in Van der Pol oscillator [21]–[22]. The oscillator is modeled as two-dimensional second order system

$$\dot{x}_1 = \dot{x}_2 \quad (1)$$

$$\dot{x}_2 = -x_1 \varepsilon h(x_1) x_2 \quad (2)$$

where

$$h(x_1) = -1 + x_1^2 \quad (3)$$

gives nonlinearity to the system. Limit cycle effect in Van der Pol oscillator can be observed by tuning the non-negative parameter ϵ , as shown by the phase portrait in Fig. 1. Note that both x_1 and x_2 are system states.

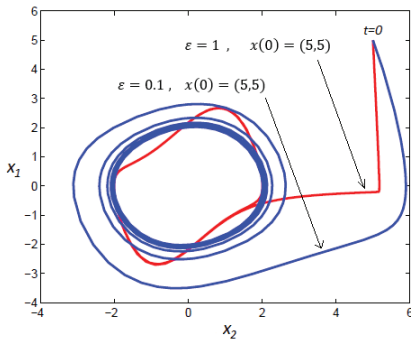


Fig. 1. Historical trajectory of x_1 and x_2

Previous literatures demonstrate methods to deal with nonlinear systems. For example, author in [18] deals with numerical second order nonlinear system with uncertainties and exogenous disturbances. The numerical nonlinear system of the form

$$\dot{x}_1 = x_2 + x_1^2 \sin 0.01t \quad (4)$$

$$\dot{x}_2 = x_1^3 e^{x_2} u + e^{x_2} \cos(x_1 x_2) \quad (5)$$

is stabilized by using advanced backstepping with nonlinear damping function. Another strange behavior of nonlinear effect can be observed in the tunnel diode [23], where the study discusses about bifurcation phenomena in tunnel diode circuit.

In linear systems, input/output frequency domain methods are known to be effective. However, in nonlinear systems, poles and zeros, frequency domain, phase and gain margin are not defined. Thus, solving nonlinear systems requires advanced control techniques. Linear systems can be stabilized by linear controllers such as pole placement approach [24]–[26], linear quadratic regulator (LQR) [27]–[29], linear quadratic Gaussian (LQG) [30], model reference adaptive control (MRAC) [7], proportional-integral-derivative (PID) technique [31]–[33] and many more. MRAC requires reference LTIV model. The error dynamic between actual model and reference model is processed by the adaptation law that can be designed by using gradient method, hyperstability or Lyapunov. Design steps for optimal control using LQR can be possible if the system matrix is linear because separation principle is valid.

II. Linearization

In some cases, linearization of nonlinear systems is normally obtainable by using the Jacobian matrix at equilibrium points [34]–[36]. Then, by using the linearized system, a simple linear controller can be applied to achieve stabilization. In some systems, linearization gives freedom to designer. For instance, designing feedback gain K for state feedback system $u(t) = -Kx(t)$ can be done by various techniques such as linear matrix inequality (LMI) and artificial computing.

Linearization via Jacobian can be the easiest technique available to linearize nonlinear function. For instance, let an autonomous system of the form

$$\frac{dx}{dt} = f(x, y) \quad (6)$$

$$\frac{dy}{dt} = g(x, y) \quad (7)$$

where $f(\cdot)$ and $g(\cdot)$ are continuously n -times differentiable C^n . Assume (x_0, y_0) be the equilibrium point for the system in equation (6) and equation (7), *Theorem 1* confirms the exponentially stable origin for (x_0, y_0) .

Theorem 1

Let $x = 0$ be an equilibrium point for the nonlinear system

$$\frac{dx}{dt} \triangleq \dot{x} = f(t, x) \quad (8)$$

where t represents time and $f: [0, \infty) \times \mathcal{D} \rightarrow \mathcal{R}^n$ is continuously differentiable, $\mathcal{D} = \{x \in \mathcal{R}^n \mid \|x\|_2 < r\}$, and the Jacobian matrix $J \triangleq \left[\frac{\partial f}{\partial x} \right]$ is bounded and Lipschitz on \mathcal{D} , uniformly in t . Let

$$A(t) = \left. \frac{\partial f}{\partial x}(t, x) \right|_{x=0} \quad (9)$$

then the origin is an exponentially stable equilibrium point for the nonlinear system if it is an exponentially stable equilibrium point for the linear system

$$\dot{x} = A(t)x \quad (10)$$

... End of Theorem 1 ...

With *Theorem 1*, the Jacobian at (x_0, y_0) can be computed as

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ &\approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) \\ &\quad + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) \end{aligned} \quad (11)$$

and

$$\begin{aligned} \frac{dy}{dt} &= g(x, y) \\ &\approx g(x_0, y_0) + \frac{\partial g}{\partial x}(x_0, y_0)(x - x_0) \\ &\quad + \frac{\partial g}{\partial y}(x_0, y_0)(y - y_0) \end{aligned} \quad (12)$$

since (x_0, y_0) is the equilibrium point, $f(x_0, y_0) \Rightarrow 0$ and, $g(x_0, y_0) \Rightarrow 0$. As such, the equilibrium points (x_0, y_0) implies

$$f(x_0, y_0) = g(x_0, y_0) = 0 \quad (13)$$

Therefore, the linearized system (6)-(7) can be concluded as in equation (14) and equation (15).

$$\frac{dx}{dt} = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) \quad (14)$$

$$\frac{dy}{dt} = \frac{\partial g}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial g}{\partial y}(x_0, y_0)(y - y_0) \quad (15)$$

Thus, Jacobian matrix J of system (6)-(7) at (x_0, y_0) can be written as

$$J = \begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) & \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0)(x - x_0) & \frac{\partial g}{\partial y}(x_0, y_0)(y - y_0) \end{bmatrix} \quad (16)$$

III. Numerical Illustration

This section illustrates two cases nonlinear systems. First, dynamical physical system will be linearized in Jacobian. Secondly, the Jacobian will be exploited to linearize the numerical nonlinear function.

A. Nonlinear Pendulum

To demonstrate the approach in equation (11) - (16), consider the dynamics of pendulum system in

$$\ddot{\theta} = -a \sin \theta - b\dot{\theta} + cT \quad (17)$$

The input to the system is the torque applied to the pendulum. In steady state, the dynamics in equation (17) appears as

$$-a \sin \theta + cT_{ss} = 0 \quad (18)$$

Then, stabilizing the pendulum at $\theta = \delta$ defines two state variables $x_1 = \theta - \delta$ and $x_2 = \dot{\theta}_n$. As such, $\dot{x}_1 = x_2$ and, $\dot{x}_2 = -a[\sin(x_1 + \delta) - \sin \delta] - bx_2 + cu$, where u represents the input (torque) to the pendulum. Hence, linearize the system at origin renders $\left. \frac{\partial x_1}{\partial x_1} \right|_{(0,0)} = 0$,

$$\left. \frac{\partial x_1}{\partial x_2} \right|_{(0,0)} = 1, \quad \left. \frac{\partial x_2}{\partial x_1} \right|_{(0,0)} = -a \cos \delta, \quad \text{and} \quad \left. \frac{\partial x_2}{\partial x_2} \right|_{(0,0)} = -b.$$

Therefore, the linearized pendulum system at origin $(0,0)$ can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a \cos \delta & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix} u \quad (19)$$

where $A = \begin{bmatrix} 0 & 1 \\ -a \cos \delta & -b \end{bmatrix}$ is the system matrix and $B = \begin{bmatrix} 0 & c \end{bmatrix}^T$ is the input matrix. The pair (A, B) is controllable. Taking $K = [K_1 \quad K_2]$ such that $A - BK$ is Hurwitz will guarantee the stability of the pendulum system. Parameter K can be formulated via state-feedback control techniques that is not within the scope of this studies.

B. Numerical Nonlinear System

Consider nonlinear system

$$\dot{x} = y \quad (20)$$

$$\dot{y} = (1 - x^2)y + x \quad (21)$$

The system is autonomous as no stimulus is exerted to the system. The nonlinearity comes from x^2y -term that appear in the 2nd-subsystem. The equilibrium point of equation (20) and equation (21) is $(x, y) = (0, 0)$. Therefore, the Jacobian matrix is formulated as

$$J = \begin{bmatrix} \frac{\partial}{\partial x}(y) & \frac{\partial}{\partial y}(y) \\ \frac{\partial}{\partial x}((1 - x^2)y + x) & \frac{\partial}{\partial y}((1 - x^2)y + x) \end{bmatrix} \quad (22)$$

and yields

$$J = \begin{bmatrix} 0 & 1 \\ -2xy + 1 & 1 - x^2 \end{bmatrix} \quad (23)$$

Then, the Jacobian matrix at equilibrium can be computed as

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad (24)$$

in order to get the linearized system at $(0,0)$ as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (25)$$

In equation (25), both x_1 and x_2 are defined as system states. As a result, and for simplicity, one would obtain

$$\dot{x} = y \quad (26)$$

$$\dot{y} = y + x \quad (27)$$

If both linearized system (equation (26) – equation (27)) and actual system (equation (20) – equation (21)) are injected by step $u(t)$, the trajectory can be recorded as in Fig. 2.

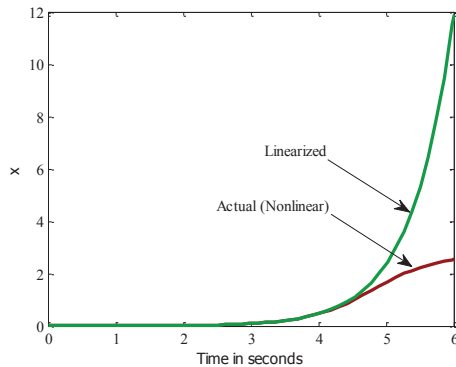


Fig. 2. Trajectory of nonlinear system and linearized system

IV. Discussion

Linearization relaxes the need of using nonlinear control techniques to stabilize nonlinear systems. However, control law that is designed by using linearized model would not be robust within the wide range of operation. This phenomenon can be realized in Fig. 2 where the trajectory of the linearized system departs away from the trajectory of the actual system when time approaching ∞ . After 4.5 seconds, the result shown in Fig. 2 bears no trajectory resemblance between actual nonlinear system in equations (20)-(21) and its linearized version in equations (26)-(27). The contradiction between the actual nonlinear system and its linearized version in Fig. 2 would devise a cunning test to control engineers. As such, the controller that is designed based on linearized model will not be able to guarantee the stabilization beyond wide range of nonlinear sector when applied to nonlinear system.

V. Conclusion

Linearize or not to linearize a nonlinear function (or system) is highly depending on the objectives of the solution to a nonlinear system under studies. Researchers studying the chaotic or bifurcation phenomena possibly will not to linearize the system at hand because relaxing nonlinear phenomena will diminish the sources of bifurcation and chaotic in the system. In some control engineering fields, linearization is a must if the nonlinearity become insignificant and does not offer catastrophic effect to the system under studies. In this case, linearization is utilize only to ease the steps for controller design.

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