Stress Intensity Factors for Crack Problems in Bonded Dissimilar Materials

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Highlights:

- The problem of inclined cracks subjected to normal and shear stress in bonded dissimilar materials was formulated.
- The modified complex potentials function method was used to formulate the hypersingular integral equations.
- The obtained system of hypersingular integral equations was solved numerically using the appropriate quadrature formula.
- The stress intensity factors at the crack tips depend on the elastic constant's ratio and crack geometries.

Abstract. The inclined crack problem in bonded dissimilar materials was considered in this study. The system of hypersingular integral equations (HSIEs) was formulated using the modified complex potentials (MCP) function method, where the continuity conditions of the resultant force and the displacement are applied. In the equations, the crack opening displacement (COD) serves as the unknown function and the traction along the cracks as the right-hand terms. By applying the curved length coordinate method and the appropriate quadrature formulas, the HSIEs are reduced to the system of linear equations. It was found that the nondimensional stress intensity factors (SIF) at the crack tips depend on the ratio of elastic constants, the crack geometries and the distance between the crack and the boundary.

Keywords: *complex variable function; bonded dissimilar materials; hypersingular integral equation; stress intensity factor.*

1 Introduction

The stress intensity factors (SIF) at the crack tip are among the physical quantities that can be used to analyze crack problems in engineering structures. Systems of

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HSIEs or singular integral equations have been proposed to find the SIF for crack problems in an infinite plane by Nik Long and Eshkuvatov in [1] and Denda and Dong in [2], and for half plane elasticity by Chen, *et al.* in [3] and Elfakhakhre, *et al.* in [4].

Crack problems in bonded dissimilar materials are discussed in [5-8]. The SIF for two inclined cracks in bonded dissimilar materials were calculated using Fredholm integral equations with the density distribution as undetermined function in [5]. The body forces method and traction free conditions of the cracks were used in finding the solution of inclined, kinked and branched cracks in bonded dissimilar materials in [6]. The nondimensional SIF for two- and three-dimensional crack problems in bonded dissimilar materials were computed using the finite element procedure based on the ratio of COD in [7]. The HSIEs were used to calculate the nondimensional SIF for multiple cracks in the upper part of bonded dissimilar materials in [8]. The mixed-mode dynamic of SIF for an interface crack in two bonded half planes was investigated by summing the extended finite element method and a domain independent interaction integrated method in [9]. The nondimensional SIF for the collinear interface cracks in two bonded half planes were calculated by combining the solution for an inner and an outer collinear crack in [10].

The objective of this paper was to determine the behavior of nondimensional SIF at the crack tips for crack problems in upper and lower parts of bonded dissimilar materials subjected to remote shear stress $\sigma_{x_1} = \sigma_{x_2} = p$ or normal stress $\sigma_{y_1} = \sigma_{y_2} = p$ by using the MCP function method.

2 Problem Formulation

The stress components $(\sigma_x, \sigma_y, \sigma_{xy})$, the resultant force function (X, Y), and the displacements (u, v) are expressed in terms of the two complex potentials $\Phi(\omega) = \phi'(\omega)$ and $\Psi(\omega) = \psi'(\omega)$ as follows:

$$\sigma_y - \sigma_x + 2i\sigma_{xy} = 2\left[\bar{\omega}\Phi'_j(\omega) + \Psi_j(\omega)\right] \tag{1}$$

$$f = -Y_j + iX_j = \phi_j(\omega) + \omega\overline{\Phi_j(\omega)} + \overline{\psi_j(\omega)} \tag{2}$$

$$2G_j(u_j + iv_j) = \kappa_j\phi_j(\omega) - \omega\overline{\Phi_j(\omega)} - \overline{\psi_j(\omega)} \tag{3}$$

where $\omega = x + iy$ is a complex variable, G_j is the shear modulus of elasticity, $\kappa_j = (3 - v_j)/(1 + v_j)$ for plane stress, $\kappa_j = 3 - 4v_j$ for plane strain, v_j is Poisson's ratio and $j = 1, 2$ [11]. The derivative of the resultant force Eq. (2) with respect to ω, yields:

$$\begin{aligned} \frac{d}{d\omega}\left\{-Y_j + iX_j\right\} &= \Phi_j(\omega) + \overline{\Phi_j(\omega)} + \frac{d\bar{\omega}}{d\omega}\left[\omega\overline{\Phi'_j(\omega)} + \overline{\Psi_j(\omega)}\right] \\ &= \left[N + iT\right]_j \end{aligned} \tag{4}$$

where the normal (N) and tangential (T) components of traction along the segment $\overline{\omega, \omega + d\omega}$ depend on the position of a point ω and the direction of the segment $d\bar{\omega}/d\omega$. The complex potentials for the crack L in an infinite plane can be expressed as [1]:

$$\phi(\omega) = \frac{1}{2\pi}\int_L \frac{g(t)\,dt}{t - \omega} \tag{5}$$

$$\psi(\omega) = \frac{1}{2\pi}\int_L \frac{\overline{g(t)}dt}{t - \omega} + \frac{1}{2\pi}\int_L \frac{g(t)\,d\bar{t}}{t - \omega} - \frac{1}{2\pi}\int_L \frac{g(t)\bar{t}dt}{(t - \omega)^2} \tag{6}$$

where $g(t)$ is COD function defined by:

$$i(\kappa + 1)g(t) = 2G\big(u(t) + iv(t)\big), t \in L \tag{7}$$

and $\big(u(t) + iv(t)\big) = \big(u(t) + iv(t)\big)^+ - \big(u(t) + iv(t)\big)^-$ denote the displacements at point t and superscript + and – are the upper and lower crack faces, respectively.

Consider two cracks L_1 and L_2 in the upper and lower parts of a bonded dissimilar material, respectively, and the conditions for remote shear stress and normal stress are:

$$\frac{1}{E_1}\sigma_{x_1} = \frac{1}{E_2}\sigma_{x_2}, \quad \frac{1}{E_1}\sigma_{y_1} = \frac{1}{E_2}\sigma_{y_2} \tag{8}$$

where $E_1 = 2G_1(1 + v_1)$ and $E_2 = 2G_2(1 + v_2)$ are Young's modulus of elasticity for upper and lower parts of bonded dissimilar materials, respectively, and assuming other stress is zero. The MCP function for crack L_1 can be described by summation of the principal $\left(\phi_{1p}(\omega),\psi_{1p}(\omega)\right)$ and complementary $\left(\phi_{1c}(\omega),\psi_{1c}(\omega)\right)$ parts of the complex potentials as follows:

$$\phi_1(\omega)=\phi_{1p}(\omega)+\phi_{1c}(\omega) \tag{9}$$

$$\psi_1(\omega)=\psi_{1p}(\omega)+\psi_{1c}(\omega) \tag{10}$$

where the principal parts of complex potentials are referred to an infinite plane elasticity. For crack L_2 the complex potentials are represented by $\phi_2(\omega)$ and $\psi_2(\omega)$. Applying continuity conditions to the resultant force Eq. (2) and displacement functions Eq. (3), then substitute Eqs. (9) and (10) and after some manipulations the following complex potentials are obtainable:

$$\phi_{1c}(\omega)=\triangle_1\left[\omega\overline{\Phi_{1p}}(\omega)+\overline{\psi_{1p}}(\omega)\right],\omega\in S_1+L_b \tag{11}$$

$$\begin{aligned}\psi_{1c}(\omega)=&\triangle_2\overline{\phi_{1p}}(\omega)-\triangle_1\Big[\omega\overline{\Phi_{1p}}(\omega)\\&+\omega^2\overline{\Phi'_{1p}}(\omega)+\omega\overline{\psi'_{1p}}(\omega)\Big],\omega\in S_1+L_b\end{aligned} \tag{12}$$

$$\phi_2(\omega)=(1+\triangle_2)\phi_{1p}(\omega),\omega\in S_2+L_b \tag{13}$$

$$\psi_2(\omega)=(\triangle_1-\triangle_2)\omega\Phi_{1p}(\omega)+(1+\triangle_1)\psi_{1p}(\omega),\omega\in S_2+L_b \tag{14}$$

where $\overline{\phi_{1p}}(\omega)=\overline{\phi_{1p}(\overline{\omega})}$, L_b is the boundary, S_1 and S_2 are the upper and lower parts of bonded dissimilar materials, respectively, and $\triangle_1,\triangle_2$ are bi-elastic constants defined as:

$$\triangle_1=\frac{G_2-G_1}{G_1+\kappa_1G_2},\ \triangle_2=\frac{\kappa_1G_2-\kappa_2G_1}{G_2+\kappa_2G_1}. \tag{15}$$

The HSIEs for the cracks in both the upper and lower parts of bonded dissimilar materials involve four traction components $\left\{N(t_0)+iT(t_0)\right\}_{jk}\ (j=1,2,k=1,2)$, which can be divided into two groups. The first two tractions $\left\{N(t_{10})+iT(t_{10})\right\}_{11}$ and $\left\{N(t_{20})+iT(t_{20})\right\}_{21}$ are obtained when the observation point is placed at points $t_{10}\in L_1$ and $t_{20}\in L_2$, respectively, caused by $g_1(t_1)$ at $t_1\in L_1$. The traction for $\left\{N(t_{10})+iT(t_{10})\right\}_{11}$ can be obtained by summing the principal and complementary parts. Substituting Eqs. (5) and (6) into Eq. (4) yields the principal part, and substituting Eqs. (11) and (12) into Eq. (4) gives the complementary part of the traction. Then, letting point ω approaches t_{10} on the crack and changing $d\bar{\omega}/d\omega$ into $d\bar{t}_{10}/dt_{10}$, yields:

$$
\begin{aligned}
&\left\{N\left(t_{10}\right)+iT\left(t_{10}\right)\right\}_{11}=\left\{N\left(t_{10}\right)+iT\left(t_{10}\right)\right\}_{1p}+\left\{N\left(t_{10}\right)+iT\left(t_{10}\right)\right\}_{1c} \\
&=\frac{1}{\pi}hp\int_{L_1}\frac{g_1\left(t_1\right)dt_1}{\left(t_1-t_{10}\right)^2}+\frac{1}{2\pi}\int_{L_1}A_1\left(t_1,t_{10}\right)g_1\left(t_1\right)dt_1+\frac{1}{2\pi}\int_{L_1}A_2\left(t_1,t_{10}\right)\overline{g_1\left(t_1\right)}dt_1
\end{aligned}
\tag{16}
$$

where

$$
\begin{aligned}
A_1\left(t_1,t_{10}\right)=B_1\left(t_1,t_{10}\right)+\triangle_1\Bigg[&\overline{B_3\left(t_1,t_{10}\right)}-\frac{d\bar{t}_{10}}{dt_{10}}\left(B_5\left(t_1,t_{10}\right)+\overline{B_4\left(t_1,t_{10}\right)}\right) \\
&+\frac{d\bar{t}_1}{dt_1}\left(B_4\left(t_1,t_{10}\right)+\overline{B_4\left(t_1,t_{10}\right)}\right)-\frac{d\bar{t}_1}{dt_1}\frac{d\bar{t}_{10}}{dt_{10}}B_6\left(t_1,t_{10}\right)\Bigg]+\triangle_2\frac{d\bar{t}_{10}}{dt_{10}}\overline{B_4\left(t_1,t_{10}\right)}
\end{aligned}
$$

$$
A_2\left(t_1,t_{10}\right)=B_2\left(t_1,t_{10}\right)+\triangle_1\left[B_4\left(t_1,t_{10}\right)+\overline{B_4\left(t_1,t_{10}\right)}-\frac{d\bar{t}_{10}}{dt_{10}}B_6\left(t_1,t_{10}\right)+\frac{d\bar{t}_1}{dt_1}B_3\left(t_1,t_{10}\right)\right]
$$

and

$$
B_1\left(t_1,t_{10}\right)=\frac{1}{\left(t_1-t_{10}\right)^2}\left(\frac{\left(t_1-t_{10}\right)^2}{\left(\bar{t}_1-\bar{t}_{10}\right)^2}\frac{d\bar{t}_1}{dt_1}\frac{d\bar{t}_{10}}{dt_{10}}-1\right)
$$

$$
B_2\left(t_1,t_{10}\right)=\frac{t_1-t_{10}}{\left(\bar{t}_1-\bar{t}_{10}\right)^3}\left[\frac{\bar{t}_1-\bar{t}_{10}}{t_1-t_{10}}\left(\frac{d\bar{t}_1}{dt_1}+\frac{d\bar{t}_{10}}{dt_{10}}\right)-2\frac{d\bar{t}_1}{dt_1}\frac{d\bar{t}_{10}}{dt_{10}}\right]
$$

$$
B_3\left(t_1,t_{10}\right)=\left[\bar{t}_1-2t_1+t_{10}\right]\frac{1}{\left(\bar{t}_1-t_{10}\right)^3}
$$

$$
B_4\left(t_1,t_{10}\right)=\frac{1}{\left(\bar{t}_1-t_{10}\right)^2}
$$

$$
B_5\left(t_1,t_{10}\right)=\frac{2\left(3\bar{t}_{10}-2t_{10}-\bar{t}_1\right)}{\left(t_1-\bar{t}_{10}\right)^3}+\frac{6\left(\bar{t}_{10}-\bar{t}_1\right)\left(\bar{t}_{10}-t_{10}\right)}{\left(t_1-\bar{t}_{10}\right)^4}
$$

$$
B_6\left(t_1,t_{10}\right)=\left[t_1+\bar{t}_{10}-2t_{10}\right]\frac{1}{\left(t_1-\bar{t}_{10}\right)^3}.
$$

Eq. (16) represents a single crack in the upper part of a bonded dissimilar material. Substituting Eqs. (13) and (14) into Eq. (4) and applying Eqs. (5) and (6) yields the traction for $\{N(t_{20})+iT(t_{20})\}_{21}$ as follows:

$$\begin{aligned}\{N(t_{20})+iT(t_{20})\}_{21} &= (1+\triangle_2)\frac{1}{\pi}\int_{L_2}\frac{g_1(t_1)dt_1}{(t_1-t_{20})^2}+\frac{1}{2\pi}\int_{L_2}A_3(t_1,t_{20})g_1(t_1)dt_1 \\ &+\frac{1}{2\pi}\int_{L_2}A_4(t_1,t_{20})\overline{g_1(t_1)}dt_1\end{aligned} \tag{17}$$

where

$$A_3(t_1,t_{20})=(1+\triangle_1)\frac{1}{\left(\bar{t}_1-\bar{t}_{20}\right)^2}\frac{d\bar{t}_1}{dt_1}\frac{d\bar{t}_{20}}{dt_{20}}-(1+\triangle_2)\frac{1}{(t_1-t_{20})^2}$$

$$A_4(t_1,t_{20})=\frac{1}{\left(\bar{t}_1-\bar{t}_{20}\right)^2}\left[(1+\triangle_2)\frac{d\bar{t}_1}{dt_1}+(1+\triangle_1)\frac{d\bar{t}_{20}}{dt_{20}}\right]$$

$$+\left[(1+\triangle_2)2t_{20}-(1+\triangle_1)2t_1+(\triangle_1-\triangle_2)\left(\bar{t}_1+\bar{t}_{20}\right)\right]\frac{1}{\left(\bar{t}_1-\bar{t}_{20}\right)^3}\frac{d\bar{t}_1}{dt_1}\frac{d\bar{t}_{20}}{dt_{20}}.$$

The second two tractions $\{N(t_{10})+iT(t_{10})\}_{12}$ and $\{N(t_{20})+iT(t_{20})\}_{22}$ are obtained when the observation points are placed at $t_{10}\in L_1$ and $t_{20}\in L_2$, respectively, caused by $g_2(t_2)$ at $t_2\in L_2$. In this process we need to introduce two bi-elastic constants defined as follows:

$$\triangle_3=\frac{G_1-G_2}{G_2+\kappa_2 G_1},\ \triangle_4=\frac{\kappa_2 G_1-\kappa_1 G_2}{G_1+\kappa_1 G_2} \tag{18}$$

which is evaluated by changing the subscript 1 to 2 and 2 to 1 in Eq. (15). The system of HSIEs for two cracks L_1 and L_2 in both the upper and the lower parts of a bonded dissimilar material is obtained as follows:

$$\{N(t_{10})+iT(t_{10})\}_1=\{N(t_{10})+iT(t_{10})\}_{11}+\{N(t_{10})+iT(t_{10})\}_{12} \tag{19}$$

$$\{N(t_{20})+iT(t_{20})\}_2=\{N(t_{20})+iT(t_{20})\}_{22}+\{N(t_{20})+iT(t_{20})\}_{21}. \tag{20}$$

In solving the system of HSIEs for a single crack in the upper part Eq. (16) and two cracks in both the upper and the lower parts Eqs. (19) and (20) of a bonded dissimilar material, it is well known that the curved length coordinate method can be used to transform the integral along the cracks into real axis s_j with an interval of $2a_j$ [1,3]. The COD function $g(t)$ is defined as follows:

$$g_j(t_j)\Big|_{t_j=t_j(s_j)} = \sqrt{a_j^2 - s_j^2}\,H_j(s_j) \tag{21}$$

where $H_j(s_j) = H_{j1}(s_j) + iH_{j2}(s_j), (j=1,2)$.

3 Results and Discussions

The SIF at the crack tips A_j and B_j of the crack $L_j\,(j=1,2)$ are defined as follows:

$$(K_1 - iK_2)_{A_j} = \sqrt{2\pi}\lim_{t\to t_{A_j}}\sqrt{|t - t_{A_j}|}\,g'_1(t_1) = \sqrt{a\pi}F_{A_j} \tag{22}$$

$$(K_1 - iK_2)_{B_j} = \sqrt{2\pi}\lim_{t\to t_{B_j}}\sqrt{|t - t_{B_j}|}\,g'_2(t_2) = \sqrt{b\pi}F_{B_j} \tag{23}$$

where $F_{A_j} = F_{1A_j} + iF_{2A_j}$ and $F_{B_j} = F_{1B_j} + iF_{2B_j}$ are the nondimensional SIF at crack tips A_j and B_j, respectively.

Consider an inclined crack with length $2R$ in the upper part of a bonded dissimilar material subjected to remote stress $\sigma_{y_1} = \sigma_{y_2} = p$ as defined in Figure 1.

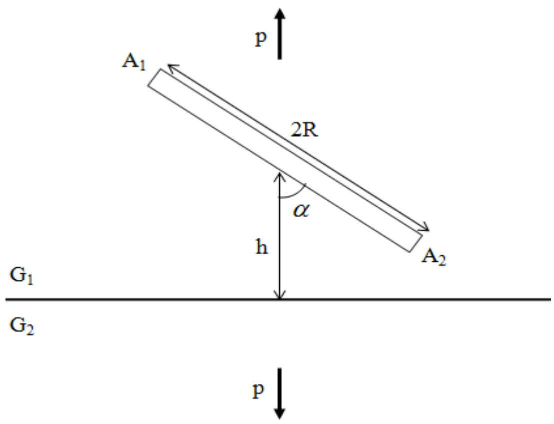


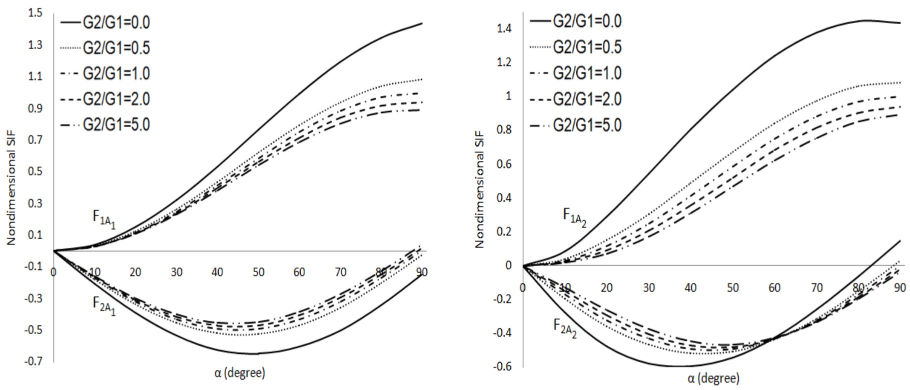
Figure 1 An inclined crack in a bonded dissimilar material.

Table 1 shows the nondimensional SIF when $\alpha = 90^o$ and $h/2R$ varies for different elastic constant ratios G_2/G_1. Our numerical results are completely in agreement with those of Isida and Noguchi [6]. It is found that the Mode I nondimensional SIF, F_1, at crack tip A_1 is equal to F_1 at tip A_2, whereas the Mode II nondimensional SIF, F_2, at crack tip A_1 is equal to the negative of F_2 at tip A_2.

Table 1 Nondimensional SIF for a Crack Parallel to the Interface (Figure 1)

G_2/G_1	SIF	$h/2R$				
		0.1	**0.2**	**0.3**	**0.4**	**0.5**
0.0	F_{1A2}	5.9498	2.9055	2.0810	1.7138	1.5110
	F_{1A2}[6]	5.9490	2.9050	2.0810	1.7140	1.5110
	F_{2A2}	3.0300	0.9940	0.4936	0.2890	0.1849
	F_{2A2}[6]	3.0310	0.9940	0.4940	0.2890	0.1850
0.5	F_{1A2}	1.1765	1.1522	1.1295	1.1083	1.0899
	F_{1A2}[6]	1.1760	1.1520	1.1300	1.1080	1.0900
	F_{2A2}	0.0950	0.0721	0.0562	0.0428	0.0322
	F_{2A2}[6]	0.0950	0.0720	0.0560	0.0430	0.0320
2.0	F_{1A2}	0.8831	0.8994	0.9125	0.9242	0.9348
	F_{1A2}[6]	0.8830	0.8990	0.9130	0.9240	0.9350
	F_{2A2}	-0.0670	-0.0489	-0.0383	-0.0302	-0.0235
	F_{2A2}[6]	-0.0670	-0.0490	-0.0380	-0.0300	-0.0240

Figure 2 shows the nondimensional SIF when $R/h = 0.9$ and α varies. It is observed that at crack tip A_1 (Figure 2(a)), as α increases F_1 increases and F_2 increases for $\alpha > 50^0$, whereas F_1 decreases and F_2 increases as G_2/G_1 increases.



(a) SIF at crack tip A_1 (b) SIF at crack tip A_2

Figure 2 SIF when $R/h = 0.9$ and α varies (Figure 1).

At crack tip A_2 (Figure 2(b)), as α increases F_1 increases and F_2 increases for $\alpha > 50^0$. As G_2/G_1 increases F_1 decreases at crack tip A_2 and, F_2 increases for $\alpha < 60^0$ and decreases for $\alpha > 60^0$.

Consider the two inclined cracks in the upper and lower parts of a bonded dissimilar material subjected to remote stress $\sigma_{x_1} = \sigma_{x_2} = p$ displayed in Figure 3.

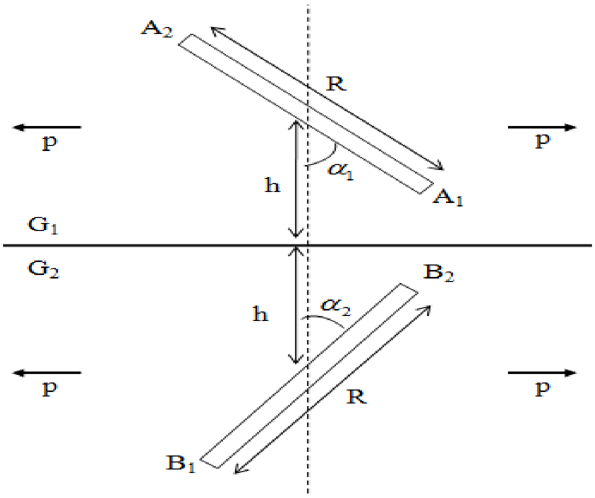


Figure 3 Two inclined cracks in a bonded dissimilar material.

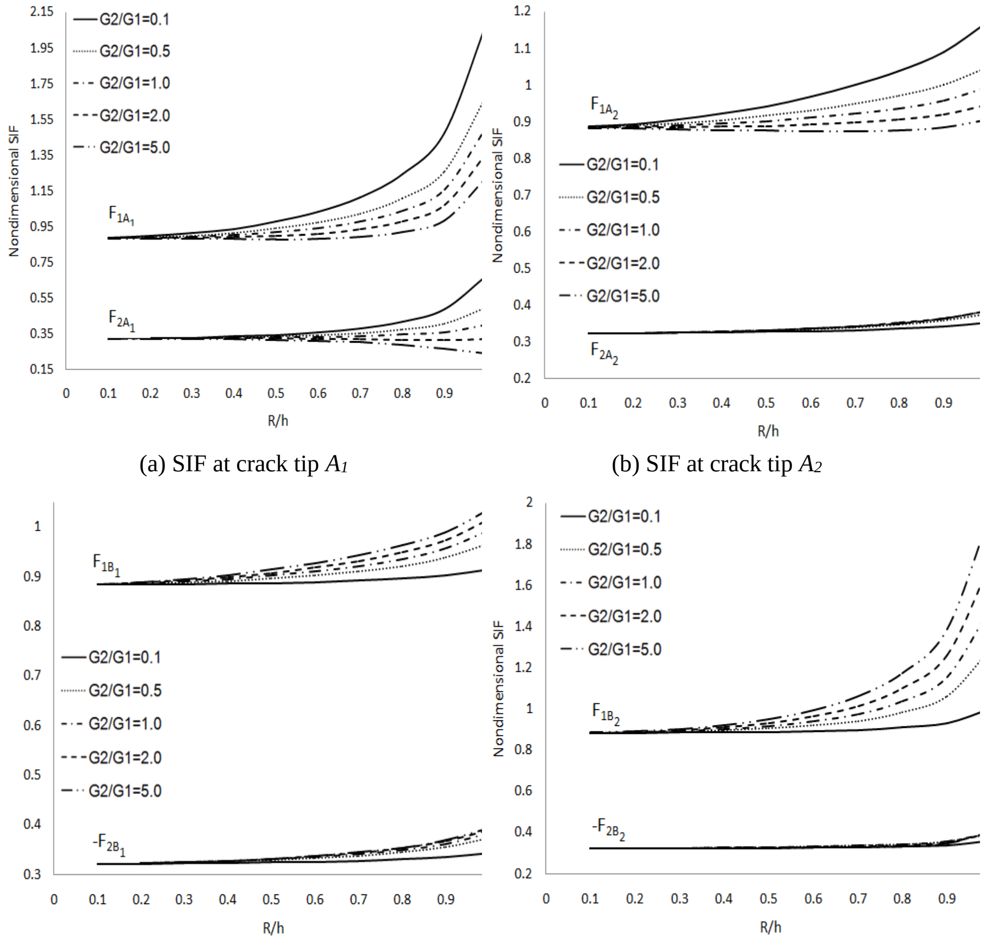
Figure 4 shows the nondimensional SIF at all cracks tips for two inclined cracks for different values of G_2/G_1 when $\alpha_1 = \alpha_2 = 20^o$ and R/h varies (Figure 3). It is observed that as R/h and G_2/G_1 increase F_1 increases at crack tips B_1 and B_2, whereas at crack tips A_1 and A_2, F_1 increases as R/h increases and F_1 decreases as G_2/G_1 increases.

As G_2/G_1 increases F_2 decreases at crack tip A_1 and increases at crack tip B_1. As R/h and G_2/G_1 increase F_2 does not show any significant difference at crack tips A_2 and B_2.

The nondimensional SIF when α_1 varies for $R/h = 0.9$ and $\alpha_2 = 45^o$ at all cracks tips are presented in Figure 5. It is found that as α_1 and G_2/G_1 increase F_1

decreases at crack tips A_1 and A_2. But F_2 increases for $\alpha_1 < 45^o$ at crack tips A_1 and A_2, and as G_2/G_1 increases F_2 decreases at crack tip A_1, and does not show any significant difference at crack tip A_2. At crack tip B_1, F_1 increases for $\alpha_1 < 20^o$ and decreases for $\alpha_1 > 20^o$, and F_1 increases as G_2/G_1 increases.

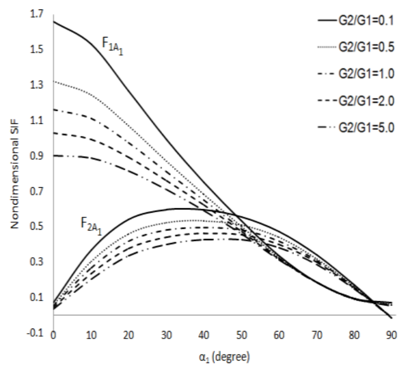
At crack tip B_2, F_1 decreases for $\alpha_1 < 40^o$ and increases as G_2/G_1 increases for $\alpha_1 < 40^o$. However F_2 does not show any significant differences at crack tips B_1 and B_2 as α_1 and G_2/G_1 increase.



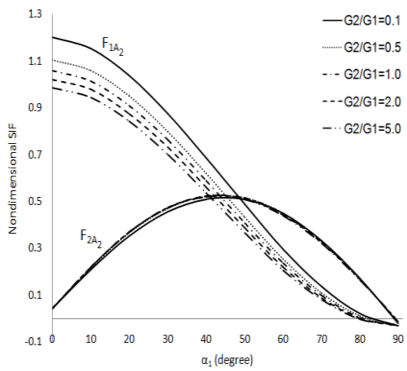
(a) SIF at crack tip A_1 (b) SIF at crack tip A_2

(c) SIF at crack tip B_1 (d) SIF at crack tip B_2

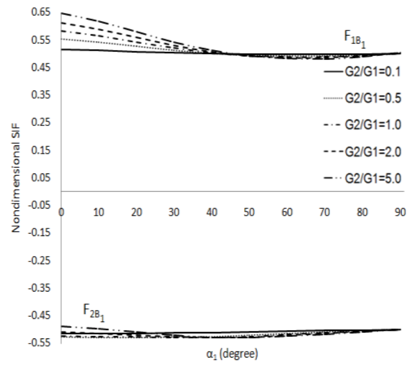
Figure 4 SIF when $\alpha_1 = \alpha_2 = 20^o$ and R/h varies (Figure 3).



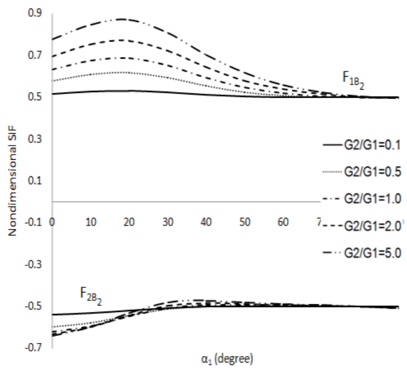
(a) SIF at crack tip A_1



(b) SIF at crack tip A_2



(c) SIF at crack tip B_1



(d) SIF at crack tip B_2

Figure 5 SIF when $\alpha_2 = 45^o$, $R/h = 0.9$ and α_1 varies (Figure 3).

4 Conclusion

An inclined crack in the upper part and two inclined cracks in the upper and lower parts of a bonded dissimilar material subjected to remote stress with different elastic constants G_1 and G_2 were studied. The systems of HSIEs for these problems were formulated by using the MCP function method. The behavior of the nondimensional SIF at all crack tips depends on the ratio of elastic constants, the crack geometries and the distance between the crack and the boundary.

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