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Multiple Solutions and Stability Analysis of Magnetic Hybrid Nanofluid Flow Over a Rotating Disk with Heat Generation

Najiyah Safwa Khashi'ie^{1,*}, Iskandar Waini¹, Nurul Amira Zainal¹, Khairum Hamzah¹, Norihan Md Arifin^{2,3}, Ioan Pop⁴

- ¹ Fakulti Teknologi Kejuruteraan Mekanikal dan Pembuatan, Universiti Teknikal Malaysia Melaka, Hang Tuah Jaya, 76100 Durian Tunggal, Melaka, Malaysia
² Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia
³ Faculty of Science, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia
⁴ Department of Mathematics, Babeş-Bolyai University, R-400084 Cluj-Napoca, Romania

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ABSTRACT

This study highlights the hybrid $\text{Fe}_3\text{O}_4\text{-CoFe}_2\text{O}_4/\text{H}_2\text{O}$ ferrofluid flow and heat transfer with the effects of linear heat generation, magnetic field and suction on a rotating disk. Using the similarity transformation, the mathematical model is simplified and reduced to a similarity set of equations. The *bvp4c* solver is used for computational analysis as well as the stability analysis procedure. The present model is successfully validated with previous results, and also verified with the fulfillment of the asymptote profiles. Triple solutions are observed within a limited range of testing parameters. The flow progress of $\text{Fe}_3\text{O}_4\text{-CoFe}_2\text{O}_4/\text{H}_2\text{O}$ is reduced when some changes made by varying the magnetic and suction parameters while a reverse result is obtained for the third solution. Only the suction parameter boosts the thermal progress of $\text{Fe}_3\text{O}_4\text{-CoFe}_2\text{O}_4/\text{H}_2\text{O}$. The stability analysis surprisingly shows that two of the solutions have positive smallest eigenvalues and align with the physical results.

1. Introduction

Ferrofluids or magnetic nanofluids have been applied in various applications such as computer drives, medicine delivery, vacuum chambers, amplifiers, revolving shaft seals and cell parting [1]. Meanwhile, hybrid ferrofluids are the combination of two solid ferromagnetic particles with a conventional heat exchange fluid like water, ethylene glycol or their mixture (water-ethylene glycol). Several studies numerically demonstrated the development of heat transfer rate in ferrofluids (see Anuar *et al.*, [2], Saranya *et al.*, [3], Waini *et al.*, [4] and Hamid *et al.*, [5]). The heat generation physically affects the distribution of temperature in the engineering applications like semiconductor wafers, electronic chips and nuclear reactors [6]. In addition, nanorefrigerant (nanofluid based on the refrigerant) is also significant in the development of heat transfer efficiency. Few studies related

* Corresponding author.

E-mail address: najiyah@utem.edu.my

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to the thermal progress of nanorefrigerant working fluid can be found in Halim and Sidik [7,8]. The unsteady flow with heat generation effect was analyzed by Zainal *et al.*, [9] specifically for the hybrid nanofluid. Besides, Elbashbeshy *et al.*, [10] showed that the Nusselt number decreased with the increment of heat generation parameter. Meanwhile, Khan *et al.*, [11] demonstrated the reduction in temperature distribution with the upsurge of heat generation factor. Very recent, Khashi'ie *et al.*, [12] analyzed and discussed the flow and thermal distributions of hybrid ferrofluid flow with thermal radiation subjected to a stretching/shrinking surface in a three-dimensional system. Many references highlighted the heat generation and thermal behaviour of the conducting fluids [13-18].

The unsteady flow has also fascinated the interest of researchers in fluid mechanics area. Vaidya *et al.*, [19,20] studied the peristaltic flow of various fluids for the application of blood flow through a narrow arteries and chyme movement, respectively. Meanwhile, other studies regarding the non-Newtonian fluids flow in a channel/tube were numerically conducted by Gudekote and Choudhari [21] and Divya *et al.*, [22] for the Casson fluid model, Gudekote *et al.*, [23] and Divya *et al.*, [24] for the Herschel-Bulkley fluid model, and Vaidya *et al.*, [25] and Gudekote *et al.*, [26] for the Rabinowitsch liquid model. Normally, the ideal steady flow suits the environment/system, however, due to few cases related to non-uniformities, fluctuations and body induction, the unsteadiness of the surrounding fluid occurs [27,28]. Wang [29] being the first to introduce the research of the unsteady flow due to a stretching sheet. Other references for the unsteady and steady flow of nanofluids can be found here [30-38].

Based on the above literature reviews, the current study is intended to carefully evaluate the multiple solutions of unsteady $\text{Fe}_3\text{O}_4\text{-CoFe}_2\text{O}_4/\text{H}_2\text{O}$ flow subjected to a rotating disk. Furthermore, the impact of heat generation on the hybrid ferrofluid flow is debatable and so the analysis of this effect is carried out. Final analysis results are portrayed in figures and tables form. These findings are significant as benchmark in the knowledge expansion regarding this prospect fluid.

2. Mathematical Formulation

Consider the $\text{Fe}_3\text{O}_4\text{-CoFe}_2\text{O}_4$ (magnetite-cobalt ferrite) flow past a rotating disk geometry with water base fluid (H_2O) as portrayed in Figure 1. The considerations for this model are

- i. The rotating velocity of the disk is $v = v_s = \Omega r / (1 - ct)$; Ω , t and c is the angular velocity (constant), time, and unsteadiness strength (constant), accordingly.
- ii. The stretching/shrinking velocity for the rotating disk is defined as $u = \lambda u_s = \lambda \Omega r / (1 - ct)$ where $\lambda > 0$ (stretching case), $\lambda = 0$ (static case) and $\lambda < 0$ (shrinking case).
- iii. The mass flux velocity of the permeable disk is $w = w_s = -w_0 / \sqrt{1 - ct}$; w_0 is a constant.
- iv. The magnetic field strength is $B^* = B_0 / \sqrt{1 - ct}$; B_0 is a constant.

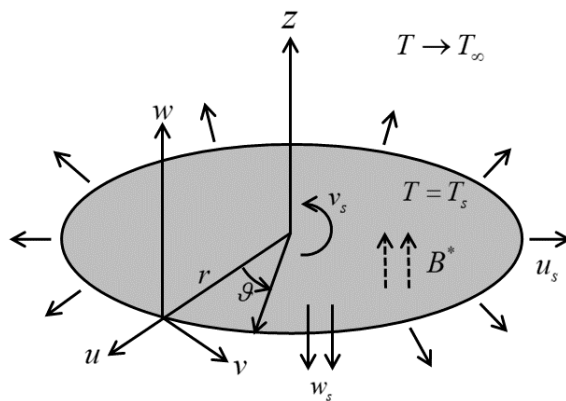


Fig. 1. The physical model

The equations which represent this two-dimensional system are given as (see Waini *et al.*, [4])

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho_{hff}} \frac{\partial p}{\partial r} + \frac{\mu_{hff}}{\rho_{hff}} \left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right) - \frac{\sigma_{hff}}{\rho_{hff}} B^{*2} u, \tag{2}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{uv}{r} = \frac{\mu_{hff}}{\rho_{hff}} \left(\frac{\partial^2 v}{\partial r^2} + \frac{\partial^2 v}{\partial z^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \right) - \frac{\sigma_{hff}}{\rho_{hff}} B^{*2} v, \tag{3}$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho_{hff}} \frac{\partial p}{\partial z} + \frac{\mu_{hff}}{\rho_{hff}} \left(\frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 w}{\partial z^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right), \tag{4}$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} = \frac{k_{hff}}{(\rho C_p)_{hff}} \left(\frac{\partial^2 T}{\partial r^2} + \frac{\partial^2 T}{\partial z^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + \frac{Q_1}{(\rho C_p)_{hff}} (T - T_\infty), \tag{5}$$

$$\begin{aligned} u = \lambda u_s, \quad v = v_s, \quad w = w_s, \quad T = T_s \quad \text{when} \quad z = 0 \\ u, v, w \rightarrow 0, \quad T \rightarrow T_\infty \quad \text{as} \quad z \rightarrow \infty \end{aligned} \tag{6}$$

The hybrid ferrofluid velocities are u, v and w , T is the temperature, $Q_1 = \frac{Q_0}{\sqrt{1-ct}}$ is the variable heat generation factor with constant Q_0 [36].

$$u = \frac{\Omega r}{1-ct} f'(\eta), \quad v = \frac{\Omega r}{1-ct} g(\eta), \quad w = -\frac{2\sqrt{\Omega \nu_f}}{\sqrt{1-ct}} f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad \eta = \sqrt{\frac{\Omega}{\nu_f}} \frac{z}{\sqrt{1-ct}}. \tag{7}$$

By substituting Eq. (7) into Eq. (2) to Eq. (6), the similarity ODEs are [4]

$$\frac{\mu_{hff}/\mu_f}{\rho_{hff}/\rho_f} f''' - \left(\frac{\sigma_{hff}/\sigma_f}{\rho_{hff}/\rho_f} M + f' \right) f' + 2ff'' - S(f' + 0.5\eta f'') + g^2 = 0, \quad (8)$$

$$\frac{\mu_{hff}/\mu_f}{\rho_{hff}/\rho_f} g'' + 2fg' - S(g + 0.5\eta g') - \left(\frac{\sigma_{hff}/\sigma_f}{\rho_{hff}/\rho_f} M + 2f' \right) g = 0, \quad (9)$$

$$\frac{1}{Pr} \frac{k_{hff}/k_f}{(\rho C_p)_{hff}/(\rho C_p)_f} \theta'' - (0.5S\eta - 2f)\theta' + \frac{Q}{(\rho C_p)_{hff}/(\rho C_p)_f} \theta = 0, \quad (10)$$

and the BCs are [4]

$$f(0) = 0.5B, \quad f'(0) = \lambda, \quad g(0) = 1, \quad \theta(0) = 1, \quad (11)$$

$$f'(\eta), g(\eta), \theta(\eta) \rightarrow 0, \quad \text{as } \eta \rightarrow \infty$$

where the involving parameters are

- i. unsteadiness parameter ($S = c/\Omega$),
- ii. suction/injection parameter ($B = w_0/\sqrt{\Omega\nu_f}$),
- iii. heat generation parameter ($Q = Q_0/(\rho C_p)_f \Omega$),
- iv. Prandtl number ($Pr = (C_p\mu)_f/k_f$), and
- v. magnetic parameter ($M = B_0^2\sigma_f/\rho_f\Omega$).

Tables 1 and Table 2 show the general correlations of hybrid nanofluid and thermophysical properties for the used nanoparticles and base fluid.

Table 1
 Properties of the water, magnetite and cobalt ferrite

| Properties | Nanoparticles | | |
|-----------------------------|---------------------|--------------------|-------------------|
| | Base fluid Water | Magnetite | Cobalt Ferrite |
| C_p (J/kgK) | 4179 | 670 | 700 |
| k (W/mK) | 0.613 | 9.8 | 3.6 |
| ρ (kg/m ³) | 997.1 | 5180 | 4908 |
| σ (S/m) | 0.05 | 0.74×10^6 | 1.1×10^7 |
| Prandtl number, (Pr) | 6.2 | - | - |

Table 2
 The hybrid ferrofluid's correlations

| Properties | Correlations |
|-------------------------|--|
| Thermal conductivity | $k_{hff} = \left[\frac{\left(\frac{\phi_1 k_1 + \phi_2 k_2}{\phi_{hff}} \right) - 2\phi_{hff} k_f + 2(\phi_1 k_1 + \phi_2 k_2) + 2k_f}{\left(\frac{\phi_1 k_1 + \phi_2 k_2}{\phi_{hff}} \right) + \phi_{hff} k_f - (\phi_1 k_1 + \phi_2 k_2) + 2k_f} \right] k_f$ |
| Electrical conductivity | $\sigma_{hff} = \left[\frac{\left(\frac{\phi_1 \sigma_1 + \phi_2 \sigma_2}{\phi_{hff}} \right) - 2\phi_{hff} \sigma_f + 2(\phi_1 \sigma_1 + \phi_2 \sigma_2) + 2\sigma_f}{\left(\frac{\phi_1 \sigma_1 + \phi_2 \sigma_2}{\phi_{hff}} \right) + \phi_{hff} \sigma_f - (\phi_1 \sigma_1 + \phi_2 \sigma_2) + 2\sigma_f} \right] \sigma_f$ |
| Heat capacity | $(\rho C_p)_{hff} = \phi_1 (\rho C_p)_{s1} + \phi_2 (\rho C_p)_{s2} + (1 - \phi_{hff}) (\rho C_p)_f$ |
| Density | $\rho_{hff} = \phi_1 \rho_{s1} + \phi_2 \rho_{s2} + (1 - \phi_{hff}) \rho_f$ |
| Dynamic viscosity | $\mu_{hff} = \frac{\mu_f}{(1 - \phi_{hff})^{2.5}}; \quad \phi_{hff} = \phi_1 + \phi_2$ |

Meanwhile, the skin friction coefficients and local Nusselt number are defined as

$$C_f = \frac{\mu_{hff}}{\rho_f u_s^2} \left(\frac{\partial u}{\partial z} \right)_{z=0}, \quad C_g = \frac{\mu_{hff}}{\rho_f v_s^2} \left(\frac{\partial v}{\partial z} \right)_{z=0}, \quad Nu_r = -\frac{rk_{hff}}{k_f (T_w - T_\infty)} \left(\frac{\partial T}{\partial z} \right)_{z=0} \quad (12)$$

Substituting Eq. (7) into Eq. (12), the following physical interests are obtained

$$\begin{aligned} Re_r^{1/2} C_f &= \frac{\mu_{hff}}{\mu_f} f''(0) \quad (\text{radial direction}), \\ Re_r^{1/2} C_g &= \frac{\mu_{hff}}{\mu_f} g'(0) \quad (\text{azimuthal direction}), \\ Re_r^{-1/2} Nu_r &= -\frac{k_{hff}}{k_f} \theta'(0), \end{aligned} \quad (13)$$

where the local Reynolds number is $Re_r = u_s r / \nu_f$.

3. Stability Analysis

There are many papers reported the stability analysis of the dual or two solutions (Wahid *et al.*, [39-41], Bakar *et al.*, [42-44], Aladdin *et al.*, [45-47] and Khashi'ie *et al.*, [48]), however, for the multiple solutions (more than two), only few of references are accessible like Waini *et al.*, [49] and Yahaya *et al.*, [50]. Following Merkin [51], Weidman *et al.*, [52] and Harris *et al.*, [53], the suitable transformation for the unsteadiness case:

$$\left. \begin{aligned} u &= \frac{\Omega r}{1-ct} \frac{\partial f}{\partial \eta}(\eta, \tau), \quad v = \frac{\Omega r}{1-ct} g(\eta, \tau), \quad w = -\frac{2\sqrt{\Omega v_f}}{\sqrt{1-ct}} f(\eta, \tau), \quad \theta(\eta, \tau) = \frac{T-T_\infty}{T_w-T_\infty}, \\ \eta &= \sqrt{\frac{\Omega}{v_f}} \frac{z}{\sqrt{1-ct}}, \quad \tau = \frac{\Omega r}{1-ct} t \text{ (time variable)} \end{aligned} \right\} \quad (14)$$

The following differential equations are obtained by substituting Eq. (14) into Eq. (2) until Eq. (5) [4]

$$\frac{\mu_{hff}/\mu_f}{\rho_{hff}/\rho_f} \frac{\partial^3 f}{\partial \eta^3} - \left(\frac{\sigma_{hff}/\sigma_f}{\rho_{hff}/\rho_f} M + \frac{\partial f}{\partial \eta} \right) \frac{\partial f}{\partial \eta} + 2f \frac{\partial^2 f}{\partial \eta^2} - S \left(\frac{\partial f}{\partial \eta} + 0.5\eta \frac{\partial^2 f}{\partial \eta^2} \right) - (1+S\tau) \frac{\partial^2 f}{\partial \eta \partial \tau} + g^2 = 0, \quad (15)$$

$$\frac{\mu_{hff}/\mu_f}{\rho_{hff}/\rho_f} \frac{\partial^2 g}{\partial \eta^2} - \left(\frac{\sigma_{hff}/\sigma_f}{\rho_{hff}/\rho_f} M + 2 \frac{\partial f}{\partial \eta} \right) g - S \left(g + 0.5\eta \frac{\partial g}{\partial \eta} \right) + 2f \frac{\partial g}{\partial \eta} - (1+S\tau) \frac{\partial^2 g}{\partial \eta \partial \tau} = 0, \quad (16)$$

$$\frac{1}{Pr} \frac{k_{hff}/k_f}{(\rho C_p)_{hff}/(\rho C_p)_f} \frac{\partial^2 \theta}{\partial \eta^2} - (0.5S\eta - 2f) \frac{\partial \theta}{\partial \eta} + \frac{Q}{(\rho C_p)_{hff}/(\rho C_p)_f} \theta - (1+S\tau) \frac{\partial \theta}{\partial \tau} = 0, \quad (17)$$

while the reduced boundary conditions are

$$\begin{aligned} f(0, \tau) &= 0.5B, \quad \frac{\partial f}{\partial \eta}(0, \tau) = \lambda, \quad g(0, \tau) = 1, \quad \theta(0, \tau) = 1, \\ \frac{\partial f}{\partial \eta}(\eta, \tau), \quad g(\eta, \tau), \quad \theta(\eta, \tau) &\rightarrow 0 \quad \text{as } \eta \rightarrow \infty. \end{aligned} \quad (18)$$

The perturbation function is designed based on Weidman *et al.*, [52]

$$\begin{aligned} f(\eta, \tau) &= f_0(\eta) + e^{-\alpha\tau} F_0(\eta, \tau), \\ g(\eta, \tau) &= g_0(\eta) + e^{-\alpha\tau} G_0(\eta, \tau), \\ \theta(\eta, \tau) &= \theta_0(\eta) + e^{-\alpha\tau} H_0(\eta, \tau) \end{aligned} \quad (19)$$

The imposition of Eq. (19) into Eq. (15) to Eq. (18) leads to the linearized equations [4,49,52]

$$\frac{\mu_{hff}/\mu_f}{\rho_{hff}/\rho_f} F_0''' - \left(\frac{\sigma_{hff}/\sigma_f}{\rho_{hff}/\rho_f} M + 2f_0' - \alpha \right) F_0' + 2(f_0 F_0'' + f_0'' F_0 + g_0 G_0) - S(F_0' + 0.5\eta F_0'') = 0, \quad (20)$$

$$\frac{\mu_{hff}/\mu_f}{\rho_{hff}/\rho_f} G_0'' - \left(\frac{\sigma_{hff}/\sigma_f}{\rho_{hff}/\rho_f} M + 2f_0' - \alpha \right) G_0 + 2(f_0 G_0' + g_0' F_0 - g_0 F_0') - S(G_0 + 0.5\eta G_0') = 0, \quad (21)$$

$$\frac{1}{Pr} \frac{k_{hff}/k_f}{(\rho C_p)_{hff}/(\rho C_p)_f} H_0'' + 2(f_0 H_0' + \theta_0' F_0) - 0.5S\eta H_0' + \left(\frac{Q}{(\rho C_p)_{hff}/(\rho C_p)_f} + \alpha \right) H_0 = 0, \quad (22)$$

$$F_0(0) = 0, F_0'(0) = 0, G_0(0) = 0, H_0(0) = 0, \quad (23)$$

$$F_0'(\eta) \rightarrow 0, G_0(\eta) \rightarrow 0, H_0(\eta) \rightarrow 0 \text{ as } \eta \rightarrow \infty.$$

Following the judgment from Harris *et al.*, [53], the boundary condition $F_0'(\infty) \rightarrow 0$ is replaced with $F_0''(0) = 1$ in order to get the smallest eigenvalue.

4. Results and Discussion

The results are obtained for the similarity solutions (Eq. (8) to Eq. (11)) and stability analysis (Eq. (20) to Eq. (23)) using the *bvp4c*. Also, the explanations of the flow and thermal behavior are discussed and presented in Figure 2 until Figure 12. Here, the validation of the current study is conducted by making a comparison with the previous studies. It is noted that the good agreement is attained between the results as given in Table 3.

The present study highlights the existence of multiple solutions; first solution, second solution and third solution based on the fulfillment of the boundary condition (see Figure 9 until Figure 12) when the parameters are used within the range of $0 \leq M \leq 0.1$ (magnetic parameter), $0 \leq Q \leq 0.1$ (heat generation parameter) and $0.2 \leq B \leq 0.3$ (suction parameter). The volumetric concentration of $Fe_3O_4-CoFe_2O_4$ is fixed as $\phi_{hff} = 2\%$ with water as the base fluid ($Pr = 6.2$) and $\lambda = 0$ is also used in the entire computation. Table 4 shows the smallest eigenvalues against suction parameter. Obviously, the first and the second solutions give the positive values which means these solutions are stable as time evolves. However, the third solution shows the opposite behavior.

Table 3

Model validation when $\phi_1 = \phi_2 = Q = M = B = 0$ and various S

| S | Present | | Waini <i>et al.</i> , [4,49] | | Fang and Tao [54] | |
|-----|-----------|----------|------------------------------|----------|-------------------|----------|
| | $g'(0)$ | $f''(0)$ | $g'(0)$ | $f''(0)$ | $g'(0)$ | $f''(0)$ |
| -1 | -0.236575 | 0.719787 | -0.2366 | 0.7198 | -0.2366 | 0.7198 |
| -2 | 0.154981 | 0.931507 | 0.1550 | 0.9315 | 0.1550 | 0.9315 |
| -5 | 1.360850 | 1.562797 | 1.3609 | 1.5628 | 1.3609 | 1.5627 |
| -10 | 3.413860 | 2.600801 | 3.4139 | 2.6008 | 3.4139 | 2.6008 |

Table 4

Smallest eigenvalues of the multiple solutions when $\phi_{hff} = 2\%$, $B = 0.2$, $M = 0.1$ and $Q = 0.05$

| S | Smallest Eigenvalues | | |
|-------|----------------------|-----------------|----------------|
| | First Solution | Second Solution | Third Solution |
| -1.11 | 5.8746 | 12.1600 | -0.6953 |
| -1.12 | 6.2075 | 12.2104 | -0.6973 |
| -1.13 | 6.4807 | 12.2178 | -0.6991 |

The impact of the magnetic parameter on the flow and thermal behaviour can be observed in Figure 2 until Figure 4. The reductions of the physical quantities ($Re_r^{1/2} C_f, Re_r^{1/2} C_g, Re_r^{-1/2} Nu_r$) for the first and the second solutions are noticed when the magnetic parameter increasing, whereas the reverse trend is shown by the third solution. From the stability analysis results in Table 4, it is concluded that, the first and second solutions are more stable than the third solution, hence the stable behaviours of the first and second solutions are taken into account. The Lorentz force is well known to be developed in the resonance of magnetic field. This force usually decelerates the fluid motion to maintain the laminar boundary layer and at the same time, develops the thermal progress of the working fluid. However, the present finding contradicts with the natural behavior of magnetic flow [49]. The unsteadiness phenomenon in fluid flow affects the natural progress of magnetic field and obstructs the enhancement of the heat transfer progress.

Apart from that, Figure 5 to Figure 7 are provided to get clear insight of suction effect on the physical quantities. Higher suction rate has the tendency to improve the values of $Re_r^{-1/2} Nu_r$, however, it reduces the values of $Re_r^{1/2} C_f, Re_r^{1/2} C_g$ for the first and the second solution. The respective profiles of the impact of suction parameter are presented in Figure 9 to Figure 11. All the solutions asymptotically fulfilled the far field boundary conditions up to boundary layer thickness (blt) $\eta_\infty = 20$, which affirms the results' validness. However, in Figure 9 to Figure 12, the blt is shown up to $\eta_\infty = 7$, so that the different profiles for all the similarity solutions can be highlighted. Besides, the influence of heat generation parameter Q on $Re_r^{-1/2} Nu_r$ is also presented in Figure 8. The imposition of Q in the boundary layer leads to decrease all the physical quantities. Meanwhile, the impact of this parameter on the temperature profile is given in Figure 12. From this figure, it is noted that the temperature is increased with the rise of Q . The profiles for the third solution shown an abnormal behaviour which proves it instability.

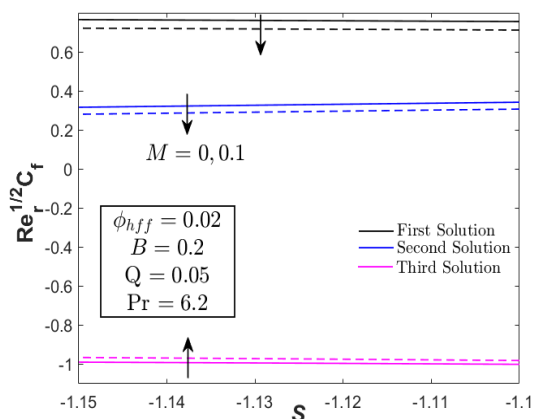


Fig. 2. $Re_r^{1/2} C_f$ for various M

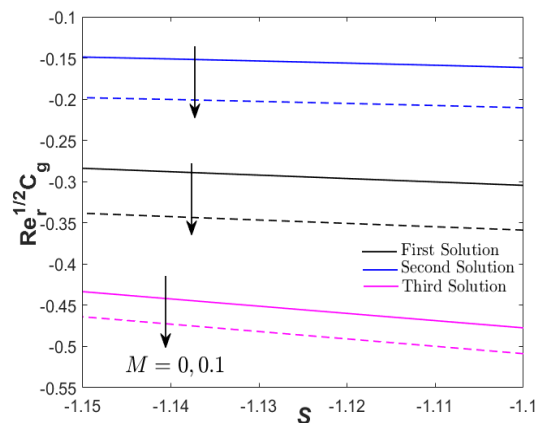


Fig. 3. $Re_r^{1/2} C_g$ for various M

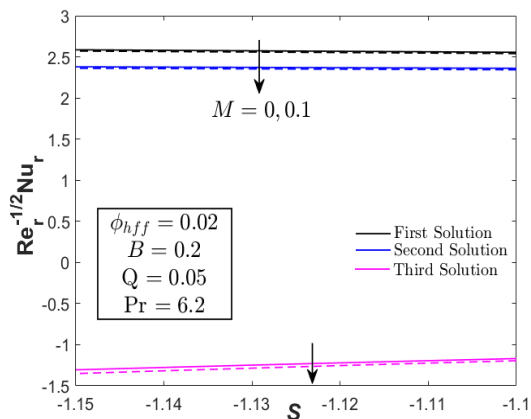


Fig. 4. $Re_r^{-1/2} Nu_r$ for various M

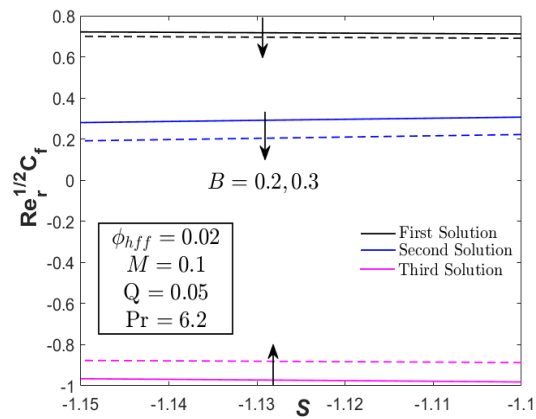


Fig. 5. $Re_r^{1/2} C_f$ for various B

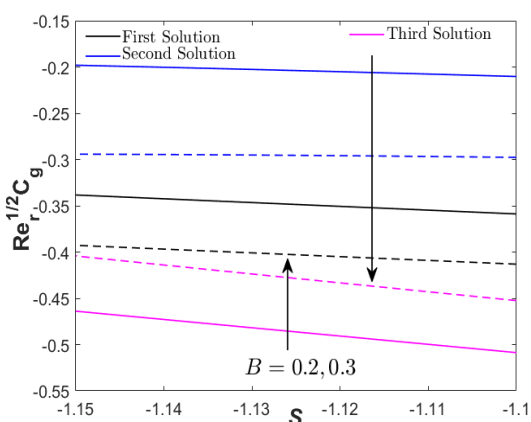


Fig. 6. $Re_r^{1/2} C_g$ for various B

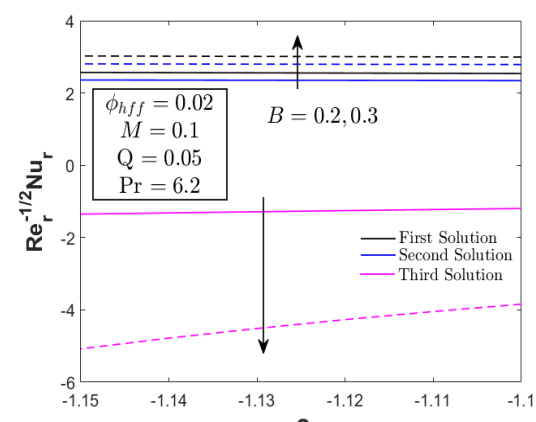


Fig. 7. $Re_r^{-1/2} Nu_r$ for various B

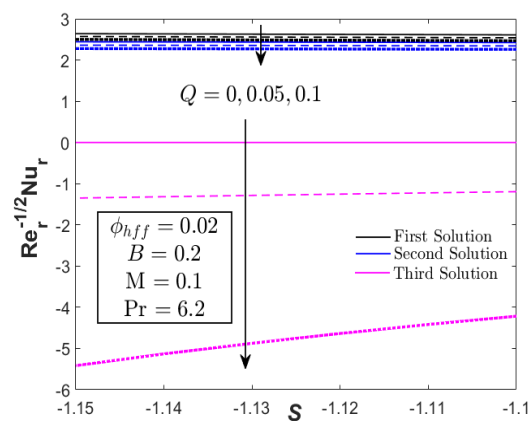


Fig. 8. $Re_r^{-1/2} Nu_r$ for several Q

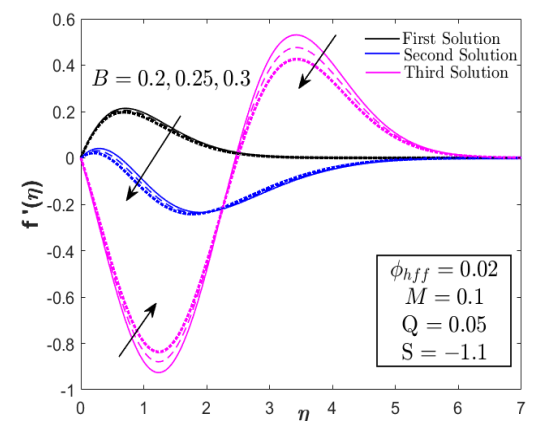


Fig. 9. Velocity (radial) with different B

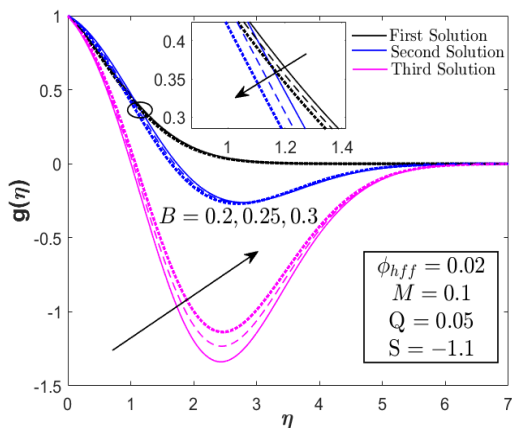


Fig. 10. Velocity (azimuthal) with different B

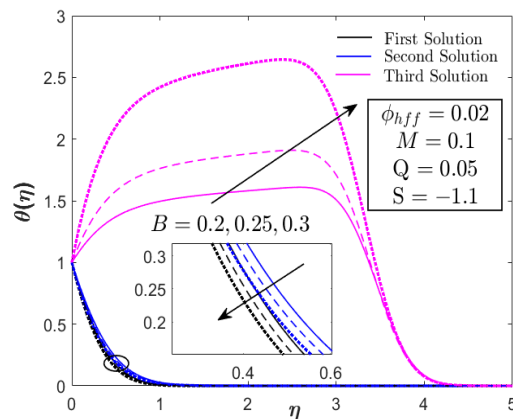


Fig. 11. Temperature with different B

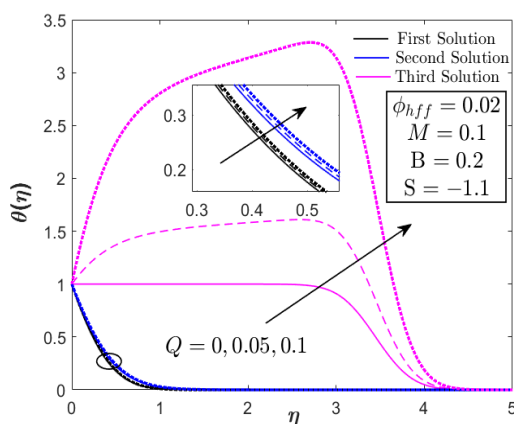


Fig. 12. Temperature with different Q

5. Conclusion

The case of unsteady flow of a magnetic hybrid nanofuid/ hybrid ferrofluid is considered subjected to a rotating disk with heat generation. The $\text{Fe}_3\text{O}_4\text{-CoFe}_2\text{O}_4$ hybrid nanoparticles are considered with water/ H_2O as the base fluid. The *bvp4c* solver is used in the generation of similarity solutions including the stability analysis procedure. The results are as follows

- i. Three solutions are obtained within the specific intervals of the physical parameters.
- ii. The first and second solutions with similar physical behaviors have positive smallest eigenvalues. Meanwhile, the third solution with a contradict flow and thermal behaviors has negative smallest eigenvalues which represents an unstable solution.
- iii. The flow behavior of $\text{Fe}_3\text{O}_4\text{-CoFe}_2\text{O}_4/\text{H}_2\text{O}$ is reduced with the increment of magnetic and suction parameters while a reverse result is obtained for the third solution.
- iv. Under the unsteadiness flow case, the heat generation parameter does not develop the flow and thermal progress.
- v. Only suction parameter boosts the thermal progress of $\text{Fe}_3\text{O}_4\text{-CoFe}_2\text{O}_4/\text{H}_2\text{O}$.

Meanwhile, the recommendation for future studies is as follows

- i. Researchers can observe other physical parameters which can develop thermal progress like thermal radiation, electromagnetohydrodynamics (EMHD) and viscous dissipation effects.
- ii. The present work only highlights the solution based on the numerical approach. The researchers also can conduct statistical data analysis for the obtained results like sensitivity analysis and response surface methodology in investigating the significant factors which contribute to the development of flow and thermal progression of the working fluid.

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