



BLASIUS FLOW OF A TERNARY HYBRID NANOFLUID OVER A MOVING PLATE

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Abstract

This study investigates the Blasius flow and heat transfer characteristics of a copper-alumina-titania/water ternary hybrid

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nanofluid over a moving flat plate. A similarity transformation is applied to reduce the governing partial differential equations into a system of similarity equations, which are then solved numerically using the Matlab-based boundary value solver. The analysis reveals the existence of dual solutions under certain flow conditions, particularly when the plate and free stream move in opposite directions. An increment of 0.5% titania concentration led to the upsurge of skin friction and heat transfer coefficients which shows that the copper-alumina-titania/water exhibits superior flow and heat transfer performances as compared to the copper-alumina/water. These findings provide a basis for optimizing operational efficiency and designing processes tailored to specific outcomes.

1. Introduction

In fluid dynamics, the study of fluid flow over a stationary plate is a well-established problem within boundary layer theory. This classical flow phenomenon was first introduced by Blasius [1]. Later, Sakiadis [2] extended this concept by analyzing fluid motion over a moving plate immersed in a quiescent medium. These foundational studies have garnered attention for their relevance to industrial processes such as plastic extrusion, continuous casting, fiber production, and crystal growth [3-8]. Hybrid nanofluids constitute an advanced category of nanofluids, formed by incorporating two (hybrid), three (ternary) or four (tetra) distinct types of nanoparticles into a base fluid. Over the past years, extensive research has been conducted to examine their applications, advantages, limitations, and thermophysical properties [9-13]. Building on these foundational works, many researchers have explored the numerical behavior of hybrid nanofluids under diverse flow and surface conditions, often utilizing property correlations proposed by Devi and Devi [14] and Takabi and Salehi [15].

Building upon the research gaps addressed by previous studies, this work investigates the thermal and flow characteristics of Blasius ternary ($\text{Al}_2\text{O}_3\text{-Cu-TiO}_2\text{/water}$) hybrid nanofluid past a moving flat plate. A key contribution of this study lies in identifying non-unique solutions for ternary nanofluid. As a preliminary investigation, these findings serve as a valuable

reference for future industrial applications. The insights gained from this study may assist engineers and scientists in determining the crucial factors and governing parameters required to achieve optimal performance in practical applications.

2. Mathematical Formulation

Consider a uniform flow denoted as u_∞ (upstream velocity) moves along a plate (x -axis direction) with uniform velocity u_w . The wall and surrounding temperatures are symbolized as T_w and T_∞ , respectively. The governing models including boundary layer and energy equations are as follows (see Khashi'ie et al. [16]):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\mu_{thnf}}{\rho_{thnf}} \frac{\partial^2 u}{\partial y^2}, \quad (2)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k_{thnf}}{(\rho C_p)_{thnf}} \frac{\partial^2 T}{\partial y^2}, \quad (3)$$

with boundary conditions (see Khashi'ie et al. [16]):

$$\begin{aligned} u &= u_w, \quad v = 0, \quad T = T_w, \quad \text{at } y = 0, \\ u &\rightarrow u_\infty, \quad T \rightarrow T_\infty, \quad \text{as } y \rightarrow \infty, \end{aligned} \quad (4)$$

where u and v are velocities aligned with x - and y - axes, respectively. Table 1 presents the thermophysical properties of both hybrid and ternary hybrid nanofluids, highlighting the concentration of alumina, copper, and titania nanoparticles, denoted as ϕ_1 , ϕ_2 and ϕ_3 , respectively. Meanwhile, Table 2 outlines the physical characteristics of the selected nanoparticles and the base fluid. To ensure the continuity equation (1) is satisfied, the following similarity variables are introduced:

$$u = u_{\infty} f'(\eta), \quad v = -\frac{1}{2} \sqrt{\frac{u_{\infty} \nu_f}{x}} [f(\eta) - \eta f'(\eta)],$$

$$\eta = y \sqrt{\frac{u_{\infty}}{\nu_f x}}, \quad \theta(\eta) = \frac{T - T_{\infty}}{T_w - T_{\infty}}. \quad (5)$$

By substituting the similarity transformations (see equation (5)) into the governing equations (2)-(4), the partial differential equations are reduced to a system of coupled nonlinear ordinary differential equations as follows:

$$\frac{\mu_{thnf}/\mu_f}{\rho_{thnf}/\rho_f} f''' + \frac{1}{2} f f'' = 0, \quad (6)$$

$$\frac{1}{Pr} \frac{k_{thnf}/k_f}{(\rho C_p)_{thnf}/(\rho C_p)_f} \theta'' + \frac{1}{2} f \theta', \quad (7)$$

$$f(0) = 0, \quad f'(0) = \lambda, \quad \theta(0) = 1,$$

$$f'(\eta) \rightarrow 1, \quad \theta(\eta) \rightarrow 0 \quad \text{as } \eta \rightarrow \infty. \quad (8)$$

The parameters in equations (6)-(8) are defined as follows: the Prandtl number ($Pr = (\mu C_p)_f / k_f$) and the velocity ratio parameter ($\lambda = u_w / u_{\infty}$).

The physical quantities of interest are the skin friction coefficient and the surface heat transfer rate. These quantities provide insight into the system's efficiency and are expressed mathematically as follows [16]:

$$Re_x^{1/2} C_f = \frac{\mu_{thnf}}{\mu_f} f''(0), \quad Re_x^{-1/2} Nu_x = -\frac{k_{thnf}}{k_f} \theta'(0), \quad Re_x = u_{\infty} x / \nu_f. \quad (9)$$

Table 1. Correlations of ternary nanofluid

Properties	
Density	$\rho_{thnf} = (1 - \phi_3) \{ (1 - \phi_2) [(1 - \phi_1) \rho_f + \phi_1 \rho_{s1}] + \phi_2 \rho_{s2} \} + \phi_3 \rho_{s3}$
Heat capacity	$(\rho C_p)_{thnf} = (1 - \phi_3) \left\{ (1 - \phi_2) \left[(1 - \phi_1) (\rho C_p)_f + \phi_1 (\rho C_p)_{s1} \right] + \phi_2 (\rho C_p)_{s2} \right\} + \phi_3 (\rho C_p)_{s3}$
Dynamic viscosity	$\frac{\mu_{thnf}}{\mu_f} = \frac{1}{(1 - \phi_1)^{2.5} (1 - \phi_2)^{2.5} (1 - \phi_3)^{2.5}}$
Thermal conductivity	$\frac{k_{thnf}}{k_{hnf}} = \left[\frac{k_{s3} + 2k_{hnf} - 2\phi_3(k_{hnf} - k_{s3})}{k_{s3} + 2k_{hnf} + \phi_3(k_{hnf} - k_{s3})} \right]$ where $\frac{k_{hnf}}{k_{nf}} = \left[\frac{k_{s2} + 2k_{nf} - 2\phi_2(k_{nf} - k_{s2})}{k_{s2} + 2k_{nf} + \phi_2(k_{nf} - k_{s2})} \right]$ $\frac{k_{nf}}{k_f} = \left[\frac{k_{s1} + 2k_f - 2\phi_1(k_f - k_{s1})}{k_{s1} + 2k_f + \phi_1(k_f - k_{s1})} \right]$

Table 2. Thermophysical properties for Cu, Al₂O₃, TiO₂ and water

Physical properties	Cu	Al ₂ O ₃	TiO ₂	Water
ρ (kg/m ³)	8933	3970	4250	997.1
C_p (J/kgK)	385	765	686.2	4179
k (W/mK)	400	40	8.9538	0.6130

3. Results and Discussion

The numerical solutions of the reduced ordinary differential equations (6)-(8) are obtained using the Matlab's bvp4c solver. This solver is based on a collocation method with adaptive mesh refinement and built-in error control. This solver operates by requiring an initial approximation that aligns with the boundary conditions set in equation (8). The first step in the computational process is coding the equations (see equations (10)-(13)) while incorporating the necessary modifications to ensure accurate numerical implementation:

$$f = y(1), \quad f' = y(2), \quad f'' = y(3), \quad \theta = y(4), \quad \theta' = y(5), \quad (10)$$

so that

$$f''' = \frac{\rho_{thnf}/\rho_f}{\mu_{thnf}/\mu_f} [-0.5y(1)y(3)], \quad (11)$$

$$\theta'' = Pr \frac{(\rho C_p)_{thnf}/(\rho C_p)_f}{k_{thnf}/k_f} [-0.5y(1)y(5)], \quad (12)$$

$$ya(1), ya(2) - \lambda, ya(4) - 1, yb(2) - 1, yb(4). \quad (13)$$

As presented in Tables 3 and 4, the current results exhibit strong agreement with previously published studies, thereby validating the reliability and accuracy of the present numerical model. This level of consistency confirms the robustness of the model and its capability to yield further reliable predictions across an extended range of governing parameters.

Table 3. Model validation of $Re_x^{1/2}C_f$ with previous studies when $\lambda = \phi_2 = \phi_3 = 0$, $Pr = 6.2$, and various ϕ_1

ϕ_1	Present	Khashi'ie et al. [16]	Ahmad et al. [17]
0	0.33206	0.33206	0.3321
0.002	0.33388	0.33388	0.3339
0.004	0.33571	0.33571	0.3357
0.008	0.33938	0.33938	0.3394
0.01	0.34123	0.34123	0.3412

Table 4. Model validation of $Re_x^{-1/2}Nu_x$ with previous studies when $\lambda = \phi_2 = \phi_3 = 0$, $Pr = 6.2$, and various ϕ_1

ϕ_1	Present	Khashi'ie et al. [16]	Ahmad et al. [17]
0	0.62007	0.62007	-
0.002	0.62241	0.62241	-
0.004	0.62475	0.62475	-
0.008	0.62943	0.62943	-
0.01	0.63177	0.63177	-

Figure 1 illustrates the variation of skin friction coefficient with different volume fractions of TiO_2 in the ternary hybrid nanofluid. The increased TiO_2

leads to a gradual enhancement in the skin friction coefficient. This trend can be attributed to the increase in the effective viscosity of the nanofluid, which is a consequence of the added solid nanoparticles. This behavior confirms the positive role of TiO_2 in intensifying wall resistance, a key factor in controlling flow characteristics in applications such as coating and extrusion processes. Meanwhile, Figure 2 demonstrates the influence of TiO_2 on the heat transfer rate at the surface. The addition of TiO_2 enhances the rate of heat transfer significantly. This enhancement is due to the superior thermal conductivity of TiO_2 as compared to the base fluid and other constituents in the nanofluid mixture. The synergistic interaction between Cu, Al_2O_3 , and TiO_2 nanoparticles results in an optimized thermal path within the ternary suspension, facilitating more efficient energy transport from the surface to the surrounding fluid. As a result, the thermal boundary layer becomes thinner, leading to a steeper temperature gradient at the surface and thus a higher heat transfer rate. Hence, it is revealed that the increasing TiO_2 content in the ternary nanofluid yields dual benefits: enhanced wall shear and superior thermal performance. These improvements suggest that such ternary nanofluids can be highly effective in thermal management applications where both momentum and heat transfer are critical, such as microelectronics cooling, energy systems, and material processing.

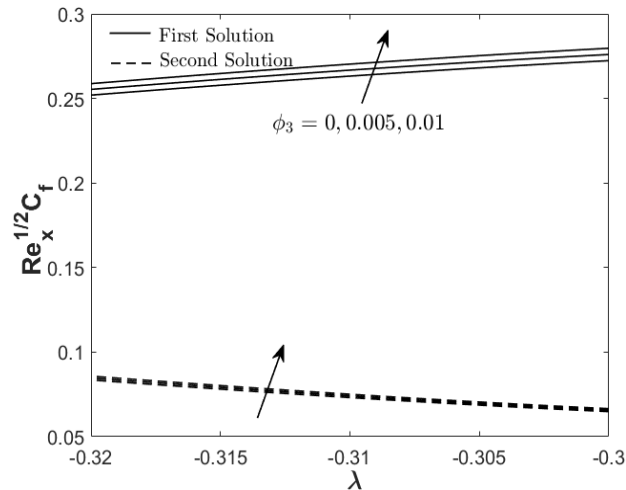


Figure 1. Effect of titania nanoparticle on the skin friction coefficient.

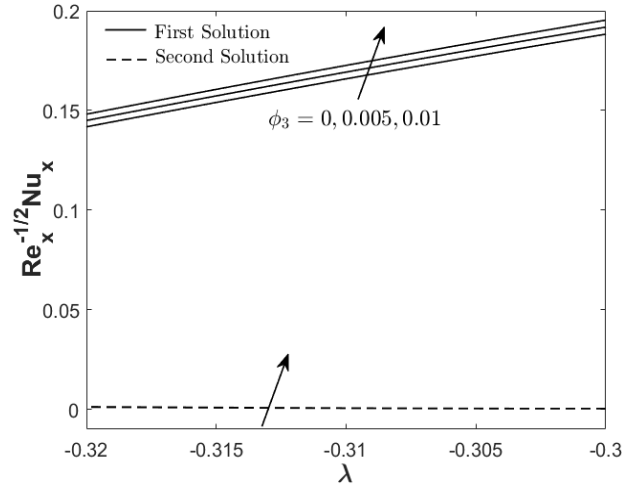


Figure 2. Effect of titania nanoparticle on the heat transfer rate.

4. Conclusions

In conclusion, this study presents a comprehensive analysis of Blasius flow involving an alumina-copper-titania ternary hybrid nanofluid over a moving plate, where the governing equations were reduced via similarity transformations and solved using Matlab's bvp4c solver. The analysis reveals the existence of dual solutions under certain flow conditions, particularly when the plate and free stream move in opposite directions. An increment of 0.5% titania concentration led to the upsurge of skin friction and heat transfer coefficients which shows that the copper-alumina-titania/water exhibits superior flow and heat transfer performances as compared to the copper-alumina/water. For future research, this study could be extended by incorporating (a) stability analysis of the multiple solutions, (b) flow and heat transfer optimization using the Response Surface Methodology (RSM) and (c) compare the heat transfer and flow performance with the tetra nanofluid.

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